When do common time series estimands have nonparametric causal meaning?*

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Abstract

The nonparametric potential outcome system provides a foundational framework for giving conditions under which common predictive time series statistical estimands, such as the impulse response function, generalized impulse response function, local projection and local projection instrument variables, have a nonparametric causal interpretation in terms of dynamic causal effects.

Keywords: causality, instrumental variable, potential outcome, prediction, shock, time series.

1 Introduction

We introduce the nonparametric *potential outcome system* as a foundational framework to give conditions under which common predictive time series statistical estimands, such as the impulse response function (Sims (1980)), generalized impulse response function (Koop et al. (1996)), local projection (Jordá (2005)) and local projection instrumental variables (Jordá et al. (2015)), have a nonparametric causal interpretation. These results are illustrated with several examples from macroeconometrics.

Quantifying dynamic causal effects is one of the great themes of the broader time series literature. Researchers use a variety of methods such as "Granger causality" (Wiener, 1956; Granger, 1969; White and Lu, 2010), highly structured models such as DSGE models (Herbst and

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Schorfheide, 2015), state space modelling (Harvey and Durbin, 1986; Harvey, 1996; Brodersen et al., 2015; Menchetti and Bojinov, 2021) as well as intervention analysis (Box and Tiao, 1975) and regression discountinuity (Kuersteiner et al., 2018). The potential outcome system is distinct. References to some of the more closely related work will be given in the next section. This paper is not focused on estimators and the associated distribution theory: we do not have much to say in that regard which is novel.

The structure of the paper is as follows. Section 2 defines the potential outcome system, which relates assignments to outcomes, allowing the corresponding dynamic causal effects to be defined. This is carried out in a way which is suitable for observational time series, refering back to some well known models in macroeconometrics. Section 3 looks at the causal meaning of common statistical estimands based on seeing the realized assignments and outcomes. The instrumented potential outcome system is defined in Section 4. It relates assignments and instruments to outcomes. In Section 5 we study the causal meaning of estimands based on seeing the realized assignments, instruments and outcomes. Section 6 looks at the causal meaning of estimands where only the instruments and outcomes are observed. Section 7 looks at the causal meaning of common statistical estimands where only the outcomes are observed. Section 8 is a Conclusion. An Appendix contains the proofs of the results given in the paper.

Notation: For a time series $\{A_t\}_{t\geq 1}$ with $A_t \in \mathcal{A}$ for all $t \geq 1$, let $A_{1:t} := (A_1, \ldots, A_t)$ and $\mathcal{A}^t := \bigtimes_{s=1}^{t} \mathcal{A}$. $A \perp\!\!\!\perp B$ says that random variables A and B are probabilistically independent.

2 The Potential Outcome System and Dynamic Causal Effects

Here we introduce the *potential outcome system*. This system extends the design-based potential outcome time series approach developed in Bojinov and Shephard (2019) to stochastic processes. We define a large class of casual estimands that summarize the dynamic causal effect of varying the assignment on future outcomes. As an illustration, we show that this system nests most, if not all, leading structural models in macroeconometrics as a special case.

The system relates to the literature on dynamic treatment effects in small-T, large-N panels. The panel work of Robins (1986) and Abbring and Heckman (2007), amongst others, led to an enormous literature on dynamic causal effects (Murphy et al., 2001; Murphy, 2003; Heckman and Navarro, 2007; Lechner, 2011; Heckman et al., 2016; Boruvka et al., 2018; Blackwell and Glynn, 2018; Hernan and Robins, 2021; Bojinov et al., 2021; Lu et al., 2017)). Beyond Bojinov and Shephard (2019), our work is closest to Angrist and Kuersteiner (2011) and Angrist et al. (2018). We will talk about their work in Section 2.3.3.

2.1 The Potential Outcome System

There is a single unit. At each time period $t \geq 1$, it receives a d_w-dimensional assignment $\{W_t\}_{t\geq 1}$. Associated with this *assignment process*, we observe a d_y-dimensional outcome $\{Y_t\}_{t\geq 1}$. The outcomes are causally related to the assignments through the potential outcome process, which describes what outcome would be observed at time t along a particular path of treatments.

Assumption 1 (Assignment and Potential Outcome)**.** *The assignment process $\{W_t\}_{t\geq 1}$ satisfies $W_t \in \mathcal{W} := \bigtimes_{k=1}^{d_w} \mathcal{W}_k \subseteq \mathcal{R}^{d_w}$. The potential outcome process is, for any deterministic sequence $\{w_s\}_{s\geq 1}$ with $w_s \in \mathcal{W}$ for all $s \geq 1$, $\{Y_t(\{w_s\}_{s\geq 1})\}_{t\geq 1}$, where the time-$t$ potential outcome satisfies $Y_t(\{w_s\}_{s\geq 1}) \in \mathcal{Y} \subseteq \mathbb{R}^{d_y}$.*

The classic case of the assignment is where $W_t = 1$ corresponds to "treatment" and $W_t = 0$ is thought of as a "control," so in that case $\mathcal{W} = \{0, 1\}$.

The potential outcome $Y_t(\{w_s\}_{s\geq 1})$ may depend on future assignments $\{w_s\}_{s\geq t+1}$. Our next assumption a priori rules this out, insisting (as one of the nine Bradford Hill (1965) criteria for causality) that causality does not work backwards in time.

Assumption 2 (Non-anticipating Potential Outcomes)**.** *For each $t \geq 1$, and all deterministic $\{w_t\}_{t\geq 1}, \{w'_t\}_{t\geq 1}$ with $w_t, w'_t \in \mathcal{W}$,*

$$Y_t(w_{1:t}, \{w_s\}_{s\geq t+1}) = Y_t(w_{1:t}, \{w'_s\}_{s\geq t+1}) \text{ almost surely.}$$

Assumption 2 is a stochastic process analogue of non-interference (Cox (1958), Rubin (1980)), extending White and Kennedy (2009) and Bojinov and Shephard (2019). Under Assumption 2, we drop references to the future assignments in the potential outcome process, and write

$$\{Y_t(\{w_s\}_{s\geq 1})\}_{t\geq 1} = \{Y_t(w_{1:t})\}_{t\geq 1}.$$

The set $\{Y_t(w_{1:t})\colon w_{1:t} \in \mathcal{W}^t\}$ collects all the potential outcomes at time t.

Together, the assignments and potential outcome generate the observable output of the system.

Assumption 3 (Output)**.** *The output is $\{W_t, Y_t\}_{t\geq 1} = \{W_t, Y_t(W_{1:t})\}_{t\geq 1}$. The $\{Y_t\}_{t\geq 1}$ is called the outcome process.*

The outcome process is the potential outcome process evaluated at the assignment process.

Even if both the assignment and outcome processes are observed, so we face the usual causal challenge. We observe the potential outcomes associated with the realized assignment path, but do not observe the potential outcomes associated with counterfactual assignment paths.

Finally, we assume that the assignment process is sequentially probabilistic, meaning that any assignment vector may be realized with positive probability at time t given the history of the observable stochastic processes up to time $t-1$. Here let $\{\mathcal{F}_t\}_{t\geq 1}$ denote the natural filtration generated by (the σ-algebra of) the realized $\{w_t, y_t\}_{t\geq 1}$.

Assumption 4 (Sequentially probabilistic assignment process)**.** *The assignment process is sequentially probabilistic, meaning for all $w \in \mathcal{W}$, $0 < P(W_t = w \mid \mathcal{F}_{t-1}) < 1$ with probability one.*

This is the time series analogue of the "overlap" condition in cross-sectional causal studies.

By putting these assumptions together, we define a potential outcome system.

Definition 1 (Potential Outcome System)**.** *Any $\{W_t, \{Y_t(w_{1:t})\colon w_{1:t} \in \mathcal{W}^t\}\}_{t\geq 1}$ satisfying Assumptions 1-4 is a* ***potential outcome system****.*

The immediate setting for a potential outcome system is where the researcher sees the assignments and the outcomes, which we will denote by

$$\{w_t^{obs}, y_t^{obs}\}_{t\geq 1}.$$

This will be the focus of the next couple of sections. Section 7 will address the case where only the outcomes are observed.

Before continuing, we highlight the generality of the potential outcome system by connecting it to several recent developments and debates in macroeconometrics.

1. The system teases out what assumptions must be placed on the assignment process to endow causal meaning to common statistical estimands without resorting to restrictive functional form assumptions.

2. A rapidly growing body of empirical research directly operate with $\{w_t^{obs}, y_t^{obs}\}_{t\geq 1}$. Empirical researchers creatively construct measures of underlying economic shocks of interest (such as monetary policy shocks, or fiscal policy shocks), and then use these constructed measures to directly estimate dynamic causal effects through reduced-form methods. This research agenda has been recently called "direct causal inference" by Nakamura and Steinsson (2018) in order to contrast it with the dominant model-based approach to causal inference in macroeconomics in the tradition of Sims (1980). We refer readers to Baek and Lee (2021) for a comprehensive review of empirical research using this direct causal inference approach.

3. The system does not rely on "invertiblitiy" or "recoverability" assumptions (Chahrour and Jurado (2021)). Understanding what can be identified about dynamic causal effects without relying on these assumptions is an active research area (Stock and Watson (2018); Plagborg-Møller (2019); Plagborg-Møller and Wolf (2020); Chahrour and Jurado (2021)).

4. The system uses no functional form restrictions. Workhorse models in macroeconometrics, such as the structural vector moving average, assume linearity. However, this nullifies state-dependence and asymmetry in dynamic causal effects. Adaptively, researchers weaken these functional form assumptions on a case-by-case basis. For example, on the possible nonlinear effects of oil prices (Killian and Vigfusson, 2011b,a; Hamilton, 2011); on the nonlinear and state dependent effects of monetary policy Tenreyro and Thwaites (2016); Jordá et al. (2020); Aruoba et al. (2021); Mavroeidis (2021), and on state-dependent fiscal multipliers (Auerbach and Gorodnichenko, 2012b,a; Ramey and Zubairy, 2018; Cloyne et al., 2020).

Remark 1 (Background processes)**.** *For the potential outcome system, we could have introduced a background process $\{X_t\}_{t\geq 1}$ that is causally unaffected by the assignment process. Such a process would play the same role as pre-assignment covariates in cross-sectional or longitudinal studies.*

2.2 Dynamic Causal Effects

In a potential outcome system, any comparison of potential outcomes at a particular point in time along different possible realizations of the assignment process defines a *dynamic causal effect*. In particular, the dynamic causal effect of the assignment process on the outcomes at time t for assignment path $w_{1:t} \in \mathcal{W}^t$ and counterfactual path $w'_{1:t} \in \mathcal{W}^t$ is $Y_t(w_{1:t}) - Y_t(w'_{1:t})$. Of course, this is an enormous class of dynamic causal effects as there are exponentially many possible paths $w_{1:t} \in \mathcal{W}^t$. Therefore, we next introduce causal estimands that average over these dynamic causal effects along various underlying assignment paths.

To do so, let us introduce some shorthand. For $t \geq 1$, $h \geq 0$, and any fixed $w \in \mathcal{W}$, write

$$Y_{t+h}(w) := Y_{t+h}(W_{1:t-1}, w, W_{t+1:t+h}),$$

the time-$(t+h)$ potential outcome at the assignment process $(W_{1:t-1}, w, W_{t+1:t+h})$. Notice that

$$Y_{t+h} = Y_{t+h}(W_t).$$

Definition 2 (Dynamic causal effects)**.** *For $t \geq 1$, $h \geq 0$, and any fixed w, w', the time-t, h-period ahead **impulse causal effect** is, $Y_{t+h}(w) - Y_{t+h}(w')$. The **filtered treatment effect** is, if it exists,*

$$\mathbb{E}\left[\{Y_{t+h}(w) - Y_{t+h}(w')\} \mid \mathcal{F}_{t-1}\right],$$

*while the corresponding **average treatment effect** is, if it exists,*

$$\mathbb{E}\left[Y_{t+h}(w) - Y_{t+h}(w')\right].$$

Here the expectations are determined by the filtered probability space of the $\{W_t, \{Y_t(w_{1:t}), w_{1:t} \in \mathcal{W}^t\}\}_{t \geq 1}$.

The impulse causal effect measures the *ceteris paribus* causal effect of intervening to switch the time-t assignment from w' to w on the h-period ahead outcomes holding all else fixed along the assignment process. This is a random object for two reasons: first, the potential outcome process itself is stochastic, and second the past $W_{1:t-1}$ and future $W_{t+1:t+h}$ assignments are stochastic.

The *filtered treatment effect*, averages the impulse causal effect conditional on the natural filtration of assignments and observed outcomes up to time $t-1$. The use of the nomenclature "filtered" in filtered treatment effect is from the stochastic process literature where filtering is the sequential estimation of a time-varying unobserved variables, e.g. Kalman filter (Kalman (1960), Durbin and Koopman (2012)), particle filter (Gordon et al. (1993), Pitt and Shephard (1999), Chopin and Papasphiliopoulos (2020)) and hidden discrete Markov models (Baum and Petrie (1966), Hamilton (1989)). This labelling fits as potential outcomes are unobserved. Note that Lee and Salanie (2020) used the title filtered treatment effect for cross-sections with partially observed assignments.

In cross-sections, it is conventional to call expected differences of potential outcomes at different values of the assignments, *average treatment effects*. This name comes from the leading case of binary assignments where $w = 1$ represents "treatment" and $w' = 0$ is stands for "control" in randomized control trial. The average treatment effects averages the filtration away from the filtered treatment effect, yielding the unconditional expectation of $Y_{t+h}(w) - Y_{t+h}(w')$.

Remark 2. *If new outcome variables were added to an existing causal study, the impulse causal effect and the average treatment effect for the existing variables would not be changed, but the filtered treatment effect might as the new outcome variables would bulk up the filtration and so possibly change the conditional expectations.*

Finally, we also define analogous versions of the dynamic causal effects for deterministic scalar treatment $w_k \in \mathcal{W}_k$, using the notation:

$$Y_{t+h}(w_k) := Y_{t+h}(W_{1:t-1}, W_{1:k-1,t}, w_k, W_{k+1:d_W,t}, W_{t+1:t+h}).$$

The corresponding time-t, h-period ahead impulse causal effect, filtered treatment effect, and average treatment effect for that specific k-th assignment are, respectively:

$$Y_{t+h}(w_k) - Y_{t+h}(w_k'), \quad \mathbb{E}[\{Y_{t+h}(w_k) - Y_{t+h}(w_k')\} \mid \mathcal{F}_{t-1}], \quad \mathbb{E}[Y_{t+h}(w_k) - Y_{t+h}(w_k')].$$

The following derivatives will appear in the properties of some important time series estimands.

Definition 3. *If they individually exists, the*

$$Y'_{t+h}(w_k) = \frac{\partial Y_{t+h}(w_k)}{\partial w_k}, \quad \mathbb{E}[Y'_{t+h}(w_k) \mid \mathcal{F}_{t-1}], \quad \mathbb{E}[Y'_{t+h}(w_k)],$$

*are called, respectively, the time-t, h-period ahead **marginal impulse causal effect**, the **marginal filtered treatment effect** and the **marginal average treatment effect**, respectively*

2.3 Examples from Macroeconomics

Here we illustrate how many leading causal models in macroeconomics can be cast as special cases of the potential outcome system, placing additional restrictions on the potential outcome process.

2.3.1 Example: The Structural Vector Moving Average Model

The structural vector moving average (SVMA) model is the leading workhorse for the study of dynamic causal effects in macroeconometrics. As emphasized in Stock and Watson (2018); Plagborg-Møller (2019); Plagborg-Møller and Wolf (2020, 2021), the SVMA model is consistent with all discrete-time dynamic stochastic general equilibrium models, and all (linear) structural vector autoregression models have an associated SVMA representation.

The SVMA model maps into the potential outcome system by assuming

$$Y_t(w_{1:t}) := \sum_{l=0}^{t-1} \Theta_l w_{t-l} + Y_t^*$$

while the associated $\{W_t\}_{t\geq 1}$ is the assignment process as defined above, $\{\Theta_l\}_{0\leq l<t}$ is a matrix of lag-coefficients and Y_t^* is a stochastic process of initial conditions.

The SVMA model imposes that the potential outcome: is linear in the assignment process and has a second source of randomness (the initial conditions Y_t^*). The SVMA is discussed in Section 7.

2.3.2 Example: Nonlinear Structural Vector Autoregressions

Some recent nonlinear structural vector autoregressions are special cases of the potential outcome system. First, consider the motivating example of Goncalves et al. (2021), which analyzes a nonlinear structural vector autoregression of the form:

$$Y_{1,t}(w_{1:t}) = w_{1,t}, \quad Y_{2,t}(w_{1:t}) = b + \beta Y_{1,t}(w_{1:t}) + \rho Y_{2,t-1}(w_{1:t-1}) + cf(Y_{1,t}(w_{1:t})) + w_{2,t},$$

where f is a nonlinear function. Given an initial condition $Y_{2,0} := \epsilon_{2,0}$, we can iterate this system of equations forward to arrive at a potential outcome process $Y_{1,t}(w_{1:t}) = w_{1,t}$, and $Y_{2,t}(w_{1:t}) =$

$g_{2,t}(w_{1:t}, \epsilon_{2,0}; \theta)$, where $g_{2,t}$ is a known function given by iteration and θ are some parameters. This is a potential outcome system where (1) $Y_{1,t}(w_{1:t})$ is non-random and only depends on the contemporaneous assignment, (2) the randomness in $Y_{2:t}(w_{1:t})$ is driven by the initial condition.

Second, consider the bivariate simultaneous equation model used to study the used to analyze the effects of monetary policy at the zero lower bound in Mavroeidis (2021) (see also Aruoba et al. (2021)). The author analyzes the system of simultaneous structural equations

$$Y_{1,t}(w_{1:t}) = \mu_1 + (\beta_1 + 1\{Y_{2,t}(w_{1:t}) = 0\}\gamma_1)w_{1,t} + (\beta_2 + 1\{Y_{2,t}(w_{1:t}) = 0\}\gamma_2)w_{2,t},$$
$$Y_{2,t}(w_{1:t}) = \max\{\mu_2 + \lambda_1 w_{1,t} + \lambda_2 w_{2,t}, 0\}.$$

Substituting the expression for $Y_{2,t}$ into $Y_{1,t}$, yields a potential outcome process

$$Y_{1,t}(w_{1:t}) = g_{1,t}(w_{1,t}, w_{2,t}; \theta), \quad Y_{2,t}(w_{1:t}) = g_{2,t}(w_{1,t}, w_{2,t}; \theta),$$

where $g_{1,t}, g_{2,t}$ are known functions given by the system and some parameters.

2.3.3 Example: Potential Outcomes of Angrist and Kuersteiner (2011), Angrist et al. (2018).

Angrist and Kuersteiner (2011) and Angrist et al. (2018) introduce a model that is also a special case of the potential outcome system. In Section 2 of Angrist and Kuersteiner (2011) sets up a system of structural equations in which (using our notation) for $t \geq 1$,

$$Y_{1,t}(w_{1:t}) = f_{1,t}(Y_{1:t-1}(w_{1:t-1}), w_{1,t}; \epsilon_0), \quad Y_{2,t}(w_{1:t}) = f_{2,t}(Y_{1,t}(w_{1:t}), w_{2,t}, w_{1:t-1}; \epsilon_0),$$

where $f_{1,t}, f_{2,t}$ are deterministic functions and ϵ_0 is a random initial condition. This imposes that $w_{1:t}$ only impacts $Y_{1,t}$ through $w_{1,t}$ directly and through $Y_{1,t-1}$ indirectly. Further, $w_{2,1:t}$ only impacts $Y_{2,t}$ contemporaneously. Related modeling and thinking includes White and Kennedy (2009); White and Lu (2010).

Through forward iteration of the system starting at $t = 1$, this can also be expressed as a potential outcome system (the iterative structure implies it satisfies non-anticipation). In particular, they defined the collection of their time-$t+h$ potential outcomes, as $\{Y_{t+h}(w^{obs}_{1:t-1}, w, W_{t+1:t+h}) : w \in \mathcal{W}_W\}$ and then their focus was on $\mathbb{E}[Y_{t+h}(w^{obs}_{1:t-1}, w, W_{t+1:t+h}) - Y_{t+h}(w^{obs}_{1:t-1}, w', W_{t+1:t+h})]$, which they called the "average policy effect."

2.3.4 Example: Expectations

Macroeconomists often consider how assignments are influenced by the distribution of future outcomes and how they in turn vary with assignments. For example, consumers and firms are modelled as forward-looking and so, expectations about future outcomes influence behavior today. A simple optimization-based version of this (e.g. Lucas (1972), Sargent (1981)) is:

$$W_t = \arg\max_{w_t} \max_{w_{t+1:T}} \mathbb{E}[\, U(Y_{t:T}(w_{1:t-1}^{obs}, w_{t:T}), w_{t:T}) \,|\, y_{1:t-1}^{obs}, w_{1:t-1}^{obs}], \tag{1}$$

where U is a utility function of future outcomes and assignments, while $\mathcal{F}_{t-1}$ is written out in long hand as $y_{1:t-1}^{obs}, w_{1:t-1}^{obs}$. For each possible $w_{t:T} \in \mathcal{W}^{T-t+1}$, the expectation is over the law of $\mathrm{Y}_{t:T}(w_{1:t-1}^{obs}, w_{t:T})|y_{1:t-1}^{obs}, w_{1:t-1}^{obs}$. This decision rule delivers $W_t, Y_t(W_{1:t})$. This looks like a potential outcome system since Assumption 2 holds. But in Equation (1), the assignment W_t is a deterministic function of past data which means it fails Assumption 4. However, incorporating any form of noise into (1) would deliver a potential outcome system.

3 Estimands Based on Assignments and Outcomes

Here we establish nonparametric conditions under which some statistical estimands based on assignments and outcomes have causal meaning in the context of an potential outcome system

$$\{W_t, \{Y_t(w_{1:t}) : w_{1:t} \in \mathcal{W}_W^t\}\}_{t \geq 1},$$

where researchers see all of

$$\{w_t^{obs}, y_t^{obs}\}_{t \geq 1},$$

the realized assignments and the realized outcomes. In particular, we ask if, for $h \geq 0$ and fixed $w_k, w_k' \in \mathcal{W}_k$, the following statistical estimands have causal meaning: impulse response function, local projection, generalized impulse response function and the local filtered projection. These estimands are defined in the middle column of Table 1. The top line results are they have the interpretation given in the right hand column of Table 1 under some important restrictions on the assignments and some additional technical conditions. The rest of this Section will spell out the details.

In this section, there is no loss in generality in assuming the outcome Y_{t+h} is univariate: the more general case is covered by running the analysis equation by equation.

3.1 Impulse Response Function

We begin by determining the conditions under which the unconditional *impulse response function* (Sims (1980)) identifies the h-period ahead average treatment effect. It is define, for deterministic $w_k, w_k' \in \mathcal{W}_k$, if it exists, by

$$IRF_{k,t,h}(w_k, w_k') := \mathbb{E}[Y_{t+h} \mid W_{k,t} = w_k] - \mathbb{E}[Y_{t+h} \mid W_{k,t} = w_k']. \tag{2}$$

Name	Estimand	Causal Interpretation
Impulse Response Function	$\mathbb{E}[Y_{t+h} \mid W_{k,t} = w_k]$ $-\mathbb{E}[Y_{t+h} \mid W_{k,t} = w_k']$	$\mathbb{E}[Y_{t+h}(w_k) - Y_{t+h}(w_k')]$;
Local Projection	$\frac{Cov(Y_{t+h}, W_{k,t})}{Var(W_{k,t})}$	$\frac{\int_{\mathcal{W}_k} \mathbb{E}[Y'_{t+h}(w_k)]\mathbb{E}[G_t(w_k)]dw_k}{\int_{\mathcal{W}_k} \mathbb{E}[G_t(w_k)]dw_k}$
Generalized Impulse Response Function	$\mathbb{E}[Y_{t+h} \mid W_{k,t} = w_k, \mathcal{F}_{t-1}]$ $-\mathbb{E}[Y_{t+h} \mid W_{k,t} = w_k', \mathcal{F}_{t-1}]$	$\text{E}[Y_{t+h}(w_k) - Y_{t+h}(w_k') \mid \mathcal{F}_{t-1}]$;
Local Filtered Projection	$\frac{\mathbb{E}[\{Y_{t+h} - \hat{Y}_{t+h\|t-1}\}\{W_{k,t} - \hat{W}_{k,t\|t-1}\}]}{\mathbb{E}[\{W_{k,t} - \hat{W}_{k,t\|t-1}\}^2]}$	$\frac{\int_{\mathcal{W}_k} \mathbb{E}\big[\mathbb{E}[Y'_{t+h}(w_k)\|\mathcal{F}_{t-1}]\mathbb{E}[G_{t\|t-1}(w_k)\|\mathcal{F}_{t-1}]\big]dw_k}{\int_{\mathcal{W}_k} \mathbb{E}[G_{t\|t-1}(w_k)]dw_k}$

Table 1: Top line results for the causal interpretation of common estimands based on assignments and outcomes. Here $G_t(w_k) = 1\{w_k \leq W_{k,t}\}(W_{k,t} - \mathbb{E}[W_{k,t}])$ and $G_{t|t-1}(w_k) = 1\{w_k \leq W_{k,t}\}(W_{k,t} - \mathbb{E}[W_{k,t} \mid \mathcal{F}_{t-1}])$, while $\hat{Y}_{t+h|t-1} = \mathbb{E}[Y_{t+h} \mid \mathcal{F}_{t-1}]$ and $\hat{W}_t = \mathbb{E}[W_t \mid \mathcal{F}_{t-1}]$. Note that $\mathbb{E}[G_t(w_k)] \geq 0$ and $\mathbb{E}[G_{t|t-1}(w_k) \mid \mathcal{F}_{t-1}] \geq 0$.

$IRF_{k,t,h}(w_k, w_k')$ can be decomposed into the average treatment effect and a selection bias term.

Theorem 1. *Assume a potential outcome system, consider some* $k = 1, \ldots, d_w$, $t \geq 1$, $h \geq 0$, *fix* $w_k, w_k' \in \mathcal{W}_k$ *and that* $\mathbb{E}[|Y_{t+h}(w_k) - Y_{t+h}(w_k')|] < \infty$. *Then,*

$$IRF_{k,t,h}(w_k, w_k') = \mathbb{E}[Y_{t+h}(w_k) - Y_{t+h}(w_k')] + \Delta_{k,t,h}(w_k, w_k'),$$

where

$$\Delta_{k,t,h}(w_k, w_k') := \frac{Cov\left(Y_{t+h}(w_k), 1\{W_{k,t} = w_k\}\right)}{\mathbb{E}[1\{W_{k,t} = w_k\}]} - \frac{Cov\left(Y_{t+h}(w_k'), 1\{W_{k,t} = w_k'\}\right)}{\mathbb{E}[1\{W_{k,t} = w_k'\}]}.$$

Proof: Given in Appendix.

Therefore, the unconditional impulse response function is equal to the average treatment effect if and only if the selection bias term

$$\Delta_{k,t,h}(w_k, w_k') = 0.$$

A sufficient condition for this to hold is that the two covariance terms are zero.

These covariance terms depend on how the assignment $W_{k,t}$ covaries with the potential outcome $Y_{t+h}(w_k)$. Since $Y_{t+h}(w_k) = Y_{t+h}(W_{1:t-1}, W_{1:k-1,t}, w_k, W_{k+1:d_W,t}, W_{t+1:t+h})$ by definition, the selection bias depends on four underlying relationships in the potential outcome process:

1. how $W_{k,t}$ relates to past assignments $W_{1:t-1}$;

2. how $W_{k,t}$ relates to other contemporaneous assignments $W_{1:k-1,t}, W_{k+1:d_W,t}$;

3. how $W_{k,t}$ relates to future assignments $W_{t+1:t+h}$;

4. how $W_{k,t}$ relates to the potential outcome process $Y_{t+h}(w_{1:t+h})$.

Theorem 2 gives sufficient conditions for $\Delta_{k,t,h}(w_k, w_k')$ to be zero.

Theorem 2. *Under the same conditions as Theorem 1, if*

$$Cov\left(Y_{t+h}(w_k), 1\{W_{k,t} = w_k\}\right) = 0, \quad Cov\left(Y_{t+h}(w_k'), 1\{W_{k,t} = w_k'\}\right) = 0. \tag{3}$$

then $\Delta_{k,t,h}(w_k, w_k') = 0$*. Moreover, (3) is satisfied if*

$$W_{k,t} \perp\!\!\!\perp Y_{t+h}(w_k), \quad \textit{and} \quad W_{k,t} \perp\!\!\!\perp Y_{t+h}(w_k'), \tag{4}$$

which is implied by

$$W_{k,t} \perp\!\!\!\perp \{Y_{t+h}(w_k) \colon w_k \in \mathcal{W}_k\}, \tag{5}$$

which is implied by

$$W_{k,t} \perp\!\!\!\perp \left(W_{1:t-1}, W_{1:k-1,t}, W_{k+1:d_W,t}, W_{t+1:t+h}, \{Y_{t+h}(w_{1:t+h}) \colon w_{1:t+h} \in \mathcal{W}^{t+h}\}\right). \tag{6}$$

Proof: Trivial, as we are assuming independence of more and more random variables. Equation (6) says the selection bias is zero if the assignment $W_{k,t}$ is randomized in the sense that it is independent of the other assignments and the time-$(t+h)$ potential outcomes.

Theorem 2 provides a nonparametric causal interpretation of a "shock" in macroeconometrics. Recent reviews on dynamic causal effects by Ramey (2016) and Stock and Watson (2018) argue intuitively that unconditional impulse response of observed outcomes to "shocks" in parametric structural models, such as the SVMA, are analogous to an average treatment effect in a randomized experiment from cross-sectional causal inference.[1] However, these statements rely on either intuitive descriptions of the statistical properties of shocks[2] or on a parametric model to link the unconditional impulse response function to an average dynamic causal effect.

Moreover, Theorems 1 and 2 clarifies a recent empirical literature that seeks to directly construct measures of the shocks of interest and measure dynamic causal effects through reduced-form estimates of unconditional impulse response functions — so called "direct causal inference." The

[1] Stock and Watson (2018) write on pg. 922: "The macroeconometric jargon for this random treatment is a 'structural shock': a primitive, unanticipated economic force, or driving impulse, that is unforecastable and uncorrelated with other shocks. The macroeconomist's shock is the microeconomists' random treatment, and impulse response functions are the causal effects of those treatments on variables of interest over time, that is, dynamic causal effects."

[2] Ramey (2016) writes on pg. 75, "the shocks should have the following characteristics: (1) they should be exogenous with respect to the other current and lagged endogenous variables in the model; (2) they should be uncorrelated with other exogenous shocks; otherwise, we cannot identify the unique causal effects of one exogenous shock relative to another; and (3) they should represent either unanticipated movements in exogenous variables or news about future movements in exogenous variables."

Theorems provide necessary and sufficient conditions that these shocks must satisfy in order for the resulting unconditional impulse response function has a non-parametric causal interpretation.

3.2 Local Projection Estimand

Under the conditions of Theorem 1, impulse response functions are causal, but estimating impulse response functions nonparametrically is not easy, particularly if $W_{k,t}$ is not discrete.

In the new literature on direct causal inference in time series, it is common for researchers to approximate the impulse response functions using the "local projection" estimator (Jordá, 2005), which directly regresses the h-step ahead outcome on the assignment of interest

$$LP_{k,t,h} := \frac{Cov(Y_{t+h}, W_{k,t})}{Var(W_{k,t})}. \tag{7}$$

What causal meaning does $LP_{k,t,h}$ have? Theorem 3 establishes that $LP_{k,t,h}$ is equivalent to a weighted average of marginal causal effects of the assignment on the h-step ahead outcome.

Theorem 3. *Under the same conditions as Theorem 1, further assume that:*

i. The support of $W_{k,t}$ is a closed interval, $\mathcal{W}_k := [\underline{w}_k, \overline{w}_k] \subset \mathbb{R}$.

ii. $Y_{t+h}(w_k)$ is continuously differentiable in w_k, as is $\mathbb{E}[Y'_{t+h}(w_k) \mid \mathcal{F}_{t-1}]$.

iii. The $W_{k,t} \perp\!\!\!\perp \{Y_{t+h}(w_k) \colon w_k \in \mathcal{W}_k\}$.

Then, it follows, if it exists, that

$$LP_{k,t,h} = \frac{\int_{\mathcal{W}_k} \mathbb{E}[Y'_{t+h}(w_k)]\mathbb{E}[G_t(w_k)]dw_k}{\int_{\mathcal{W}_k} \mathbb{E}[G_t(w_k)]dw_k},$$

where $G_t(w_k) = 1\{w_k \leq W_{k,t}\}(W_{k,t} - \mathbb{E}[W_{k,t}])$, noting $\mathbb{E}[G_t(w_k)] \geq 0$;

Proof: Given in Appendix.

This says that $LP_{k,t,h}$ is a weighted average of marginal average treatment effects of $W_{k,t}$ on the Y_{t+h}, where

$$\mathbb{E}[G_t(w_k)] = (\mathbb{E}[W_{k,t}|W_{k,t} > w_k] - \mathbb{E}[W_{k,t}])\, P(W_{k,t} > w_k)$$

and so the weights are non-negative and sum to one. Thus, if the $W_{k,t}$ assignment is a shock in the sense stated in Theorem 2, the local projection estimand is a nonparametric causal estimand.

3.3 Generalized Impulse Response Function

In non-linear time series models, it is common for researchers to focus on the conditional version of the impulse response function, the h-period ahead *generalized impulse response function* (Gallant et al. (1993); Koop et al. (1996); Gourieroux and Jasiak (2005)), which is

$$GIRF_{k,t,h}(w_k, w_k' \mid \mathcal{F}_{t-1}) := \mathbb{E}[Y_{t+h} \mid W_{k,t} = w_k, \mathcal{F}_{t-1}] - \mathbb{E}[Y_{t+h} \mid W_{k,t} = w_k', \mathcal{F}_{t-1}]. \quad (8)$$

$GIRF_{k,t,h}$ decomposes into the filtered treatment effect and a selection bias term.

Theorem 4. *Assume a potential outcome system, some $k = 1, \ldots, d_w$, $t \geq 1$, and $h \geq 0$ and that $\mathbb{E}[|Y_{t+h}(w_k) - Y_{t+h}(w_k')| \mid \mathcal{F}_{t-1}] < \infty$. Then, for any deterministic $w_k, w_k' \in \mathcal{W}$,*

$$GIRF_{k,t,h}(w_k, w_k' \mid \mathcal{F}_{t-1}) = \mathbb{E}[\{Y_{t+h}(w_k) - Y_{t+h}(w_k')\} \mid \mathcal{F}_{t-1}] + \Delta_{k,t,h}(w_k, w_k' \mid \mathcal{F}_{t-1}),$$

where

$$\Delta_{k,t,h}(w_k, w_k' \mid \mathcal{F}_{t-1}) := \frac{Cov\left(Y_{t+h}(w_k), 1\{W_{k,t} = w_k\} \mid \mathcal{F}_{t-1}\right)}{\mathbb{E}[1\{W_{k,t} = w_k\} \mid \mathcal{F}_{t-1}]} - \frac{Cov\left(Y_{t+h}(w_k'), 1\{W_{k,t} = w_k'\} \mid \mathcal{F}_{t-1}\right)}{\mathbb{E}[1\{W_{k,t} = w_k'\} \mid \mathcal{F}_{t-1}]}.$$

Proof: Given in Appendix.

Sufficient conditions for the selection bias term $\Delta_{k,t,h}(w_k, w_k' \mid \mathcal{F}_{t-1})$ to equal zero is that the two conditional covariances are zero. Repeating the unconditional case, Theorem 5 provides sufficient conditions such that the selection bias term is equal to zero.

Theorem 5. *Under the same conditions as Theorem 4, if*

$$Cov\left(Y_{t+h}(w_k), 1\{W_{k,t} = w_k\} \mid \mathcal{F}_{t-1}\right) = 0, \quad Cov\left(Y_{t+h}(w_k'), 1\{W_{k,t} = w_k'\} \mid \mathcal{F}_{t-1}\right) = 0, \quad (9)$$

then $\Delta_{k,t,h}(w_k, w_k') = 0$. Moreover, (9) is implied by

$$W_{k,t} \perp\!\!\!\perp Y_{t+h}(w_k) \mid \mathcal{F}_{t-1}, \quad \textit{and} \quad W_{k,t} \perp\!\!\!\perp Y_{t+h}(w_k') \mid \mathcal{F}_{t-1}, \quad (10)$$

which is implied by

$$[W_{k,t} \perp\!\!\!\perp \{Y_{t+h}(w_k) \colon w_k \in \mathcal{W}_k\}] \mid \mathcal{F}_{t-1}, \quad (11)$$

which is implied by

$$[W_{k,t} \perp\!\!\!\perp \left(W_{1:k-1,t}, W_{k+1:d_W,t}, W_{t+1:t+h}, \{Y_{t+h}(w_{1:t-1}^{obs}, w_{t:t+h}) \colon w_{t:t+h} \in \mathcal{W}^{h+1}\}\right)] \mid \mathcal{F}_{t-1}. \quad (12)$$

Proof: Trivial.

Therefore, under (9), the selection bias $\Delta_{k,t,h}(w_k, w'_k \mid \mathcal{F}_{t-1}) = 0$ and the conditional impulse response function identifies the filtered impulse causal effect. Notice how much weaker (12) is than (6), as it allows the assignment to probabilistically depend flexibly on the past realised potential outcomes and realised assignments — but the assignment needs to be independent of all future and other contemporaneous assignments, as well as the future potential outcomes.

At first glance, (11) appears analogous to a typical unconfoundedness from cross-sectional causal inference or sequential randomization assumption from longitudinal causal inference. That is, it imposes that conditional on the history up to time $t-1$, the assignment $W_{k,t}$ must be as good as randomly assigned. However, recall that the notation $Y_{t+h}(w_k)$ buries dependence on (i) other contemporaneous assignments $W_{1:k-1,t}, W_{k+1:d_W,t}$; (ii) future assignments $W_{t+1:t+h}$; and (iii) the potential outcomes at time-$(t+h)$. Therefore, Theorem 5 provides in equation (12) further sufficient conditions under which (11) is satisfied, highlighting that it is sufficient to further impose that the assignment $W_{k,t}$ is jointly independent of all other contemporaneous and future assignments as well as the underlying potential outcomes.

Remark 3. *How do the conditions in Theorem 2 relate to the conditions in Theorem 5?*

Applying the law of total covariance yields

$$Cov(Y_{t+h}(w_k), 1\{W_{k,t} = w_k\}) = \mathbb{E}[Cov(Y_{t+h}(w_k), 1\{W_{k,t} = w_k\} \mid \mathcal{F}_{t-1})] \tag{13}$$

$$+ Cov(\mathbb{E}[Y_{t+h}(w_k) \mid \mathcal{F}_{t-1}], \mathbb{E}[1\{W_{k,t} = w_k\} \mid \mathcal{F}_{t-1}]), \tag{14}$$

so $Cov(Y_{t+h}(w_k), 1\{W_{k,t} = w_k\}) = 0$ *neither implies or is implied by* $Cov(Y_{t+h}(w_k), 1\{W_{k,t} = w_k\} \mid \mathcal{F}_{t-1})] = 0$*. Hence, the conditional and unconditional cases are non-nested.*

If we work probabilistically then the condition

$$W_{k,t} \perp\!\!\!\perp \left(W_{1:t-1}, W_{1:k-1,t}, W_{k+1:d_W,t}, W_{t+1:t+h}, \{Y_{1:t+h}(w_{1:t+h}) \colon w_{1:t+h} \in \mathcal{W}^{t+h}\}\right)$$

does imply the condition

$$[W_{k,t} \perp\!\!\!\perp \left(W_{1:k-1,t}, W_{k+1:d_W,t}, W_{t+1:t+h}, \{Y_{t+h}(w^{obs}_{1:t-1}, w_{t:t+h}) \colon w_{t:t+h} \in \mathcal{W}^{h+1}\}\right)] \mid \mathcal{F}_{t-1}.$$

This second point is important practically. The generalized impulse response function, tells us the filtered treatment effect, under $[W_{k,t} \perp\!\!\!\perp \{Y_{t+h}(w_k) \colon w_k \in \mathcal{W}_k\}] \mid \mathcal{F}_{t-1}$*. So a temporally averaged generalized impulse response function tells us the average treatment effect without the need to employ the harsher condition* $[W_{k,t} \perp\!\!\!\perp \{Y_{t+h}(w_k) \colon w_k \in \mathcal{W}_k\}]$ *as it sidesteps the use of the impulse response function.*

3.4 Local Filtered Projection Estimand

Again, estimating the generalized impulse response functions nonparametrically is not easy. Under the same conditions as Theorem 3 but replacing condition iii with equation (11), the filtered local projection

$$\frac{Cov(Y_{t+h}, W_{k,t} \mid \mathcal{F}_{t-1})}{Var(W_{k,t} \mid \mathcal{F}_{t-1})}$$

equals

$$\frac{\int_{\mathcal{W}_k} \mathbb{E}[Y'_{t+h}(w_k) \mid \mathcal{F}_{t-1}] \mathbb{E}[G_{t|t-1}(w_k) \mid \mathcal{F}_{t-1}] dw_k}{\int_{\mathcal{W}_k} \mathbb{E}[G_{t|t-1}(w_k) \mid \mathcal{F}_{t-1}] dw_k},$$

where $G_{t|t-1}(w_k) = 1\{w_k \leq W_{k,t}\}(W_{k,t} - \mathbb{E}[W_{k,t} \mid \mathcal{F}_{t-1}])$, noting $\mathbb{E}[G_{t|t-1}(w_k) \mid \mathcal{F}_{t-1}] \geq 0$. That is, the filtered local projection is equivalent to a weighted average of conditional average marginal effects of $W_{k,t}$ on Y_{t+h}, where the weights now depend on the natural filtration but still are non-negative and sum to one.

Of more practice importance, is the local projection of $Y_{t+h} - \hat{Y}_{t+h|t-1}$ on $W_{k,t} - \hat{W}_{k,t|t-1}$. We call this local filtered projection and it is defined as

$$\frac{\mathbb{E}[\{Y_{t+h} - \hat{Y}_{t+h|t-1}\}\{W_{k,t} - \hat{W}_{k,t|t-1}\}]}{\mathbb{E}[\{W_{k,t} - \hat{W}_{k,t|t-1}\}^2]}$$

which equals, under the same conditions as needed for the filtered local projection plus needing the unconditional expectations to exist,

$$\frac{\int_{\mathcal{W}_k} \mathbb{E}\left[\mathbb{E}[Y'_{t+h}(w_k) \mid \mathcal{F}_{t-1}] \mathbb{E}[G_{t|t-1}(w_k) \mid \mathcal{F}_{t-1}]\right] dw_k}{\int_{\mathcal{W}_k} \mathbb{E}[G_{t|t-1}(w_k)] dw_k},$$

This is a long-run weighted average of the marginal filtered causal effect. The weights are non-zero and average to one over time.

An alternative to the generalized local projection is to replace the conditional expectations $\hat{Y}_{t+h|t-1}$ and $\hat{W}_{t+h|t-1}$ with best linear forecasts, based on the filtration.

4 The Instrumented Potential Outcome System

We now use a special case of the potential outcome system to encompass instrumental variables. The instrumented potential outcome system will give instrument variable type statistics nonparametric causal meaning. A rapidly growing literature in macroeconomics exploits the use of instruments to identify dynamic causal effects (Jordá et al. (2015), Gertler and Karadi (2015), Stock and Watson (2018), Plagborg-Møller and Wolf (2020), Jordá et al. (2020), Baek and Lee (2021) and many others.) Section 5 details the case where researchers see the assignments, the instruments and the outcomes. Section 6 looks at where only the instruments and the outcomes are seen.

This work is closest to Jordá et al. (2020), who used a type of potential outcome structure to understand the causal content of local projection-IV, relating it to the LATE interpretation of instrumental variable regression developed in cross-sectional econometrics (Imbens and Angrist (1994); Angrist et al. (1996)).

4.1 The Instrumented System

We start by setting up an artificial or "augmented assignment" V_t, so that

$$\left\{V_t, \left\{Y_t(v_{1:t}) : v_{1:t} \in \mathcal{W}_V^t\right\}\right\}_{t\geq 1}$$

is assumed to be a potential outcome system. So far, there is nothing new.

The instrumented potential outcome system imposes two constraints on the potential outcome system: (i) that $\{V_t\}_{t\geq 1}$ splits into an "instrument" $\{Z_t\}_{t\geq 1}$ and the "potential assignment" $\{W_t(z_t) : z_t \in \mathcal{W}_Z^t\}_{t\geq 1}$, which is only moved by the contemporaneous instrument, so $V_t = \{Z_t, W_t(Z_t)\}$; (ii) the potential outcomes are only moved by the assignment $W_{1:t}$, in the usual way, not all of $V_{1:t}$. Definition 4 is a formal statement of this.

Definition 4 (Instrumented potential outcome system)**.** *Assume $W_t \in \mathcal{W}_W$, $Z_t \in \mathcal{W}_Z$ and write $V_t = (W_t, Z_t)$. Assume $\left\{V_t, \left\{Y_t(v_{1:t}) : v_{1:t} \in (\mathcal{W}_W \times \mathcal{W}_Z)^t\right\}\right\}_{t\geq 1}$ is a potential outcome system. Additionally, enforce three Assumptions:*

i. The k-th "potential assignment",

$$\left\{W_{k,t}(\{z_s\}_{s\geq 1})\right\}_{t\geq 1} = \left\{W_{k,t}(z'_{1:t-1}, z_t, \{z'_s\}_{s\geq t+1})\right\}_{t\geq 1},$$

almost surely, for all deterministic $\{z_t\}_{t\geq 1}$, and $\{z'_t\}_{t\geq 1}$. For the other assignments: $W_{1:k-1,t}(\{z_s\}_{s\geq 1}) = W_{1:k-1,t}(\{z'_s\}_{s\geq 1})$, and $W_{k+1:d_W,t}(\{z_s\}_{s\geq 1}) = W_{k+1:d_W,t}(\{z'_s\}_{s\geq 1})$ almost surely, for all deterministic $\{z_t\}_{s\geq 1}$ and $\{z'_t\}_{t\geq 1}$. Write the potential assignments as

$$\{W_t(z_t) = (W_{1:k-1,t}, W_{k,t}(z_t), W_{k+1:d_W,t}) : z_t \in \mathcal{W}_Z\},$$

while the assignment is

$$W_t = W_t(Z_t) = (W_{1:k-1,t}, W_{k,t}(Z_t), W_{k+1:d_W,t}).$$

ii. That

$$Y_t((w_1, z_1), ..., (w_t, z_t)) = Y_t((w_1, z'_1), ..., (w_t, z'_t))$$

for all $w_{1:t} \in \mathcal{W}_W^t$ and $z_{1:t}, z'_{1:t} \in \mathcal{W}_Z^t$. Write the potential outcomes as $\{Y_t(w_{1:t}) : w_{1:t} \in \mathcal{W}_W^t\}$ and outcome as $Y_t = Y_t(W_{1:t})$.

iii. The output is

$$\{Z_t, W_t, Y_t\}_{t\geq 1} = \{Z_t, W_t(Z_t), Y_t(W_{1:t})\}_{t\geq 1},$$

while Z_t and $\{Z_t\}_{t\geq 1}$ are called the "contemporaneous instrument," and instrument process, respectively. The realised output is observed and sometimes written as $\{z_t^{obs}, w_t^{obs}, y_t^{obs}\}_{t\geq 1}$.

Any

$$\{Z_t, \{W_t(z_t), z_t \in \mathcal{W}_Z\}, \{Y_t(w_{1:t}), w_{1:t} \in \mathcal{W}_W^t\}\}_{t\geq 1}$$

satisfying (a)-(c) is an **instrumented potential outcome system**.

The classic case of this setup in cross-sectional studies in the presence of noncompliance. There the instrument $Z_t = 1$ corresponds to "intention to treat" and $Z_t = 0$ is "intention to control", with $\mathcal{W}_W \in \{0, 1\}$ and $\mathcal{W}_Z \in \{0, 1\}$. Then when $W_t(1) = 1$, there is treatment as intended while $W_t(0) = 0$ corresponds to control as intended. But there can be noncompliance, when $W_t(1) = 0$ and $W_t(0) = 1$.

Assumption (ii) is the familiar outcome exclusion restriction on the instrument from cross-sectional causal inference. Assumption (i) imposes that Z_t is only an instrument for the time-t, k-th assignment. This formalizes common empirical intuition in macroeconometrics where a constructed external instrument is often "targeted" towards a single economic shock of interest (e.g., researchers construct proxies for the monetary policy shock or fiscal policy shock).

To use this structure we also need a type of "relevance" condition. Such conditions will be stated when it is used below.

5 Estimands Based on Assignments, Instruments and Outcomes

Here we study the nonparametric conditions under which some statistical estimands based on assignments, instruments and outcomes have causal meaning in the context of an instrumented potential outcome system

$$\{Z_t, \{W_t(z_t), z_t \in \mathcal{W}_Z\}, \{Y_t(w_{1:t}), w_{1:t} \in \mathcal{W}_W^t\}\}_{t\geq 1},$$

where researchers see the instruments, the assignments and the outcomes

$$\{z_t^{obs}, w_t^{obs}, y_t^{obs}\}_{t\geq 1}.$$

Here the assignments themselves are directly observable, and so the researcher constructs a dynamic IV estimand that takes the ratio of an impulse response function of the outcome on the

Name	Estimand	Causal Interpretation
Wald	$\frac{\mathbb{E}[Y_{t+h} \mid Z_t=z]-\mathbb{E}[Y_{t+h} \mid Z_t=z']}{\mathbb{E}[W_{k,t} \mid Z_t=z]-\mathbb{E}[W_{k,t} \mid Z_t=z']}$	$\frac{\int_{\mathcal{W}} \mathbb{E}[Y'_{t+h}(w_k) \mid H_t(w_k)=1]\mathbb{E}[H_t(w_k)]dw_k}{\int_{\mathcal{W}} \mathbb{E}[H_t(w_k)]dw_k}$
IV	$\frac{Cov(Y_{t+h},Z_t)}{Cov(W_t,Z_t)}$	$\frac{\int_{\mathcal{W}_Z} \mathbb{E}[Y'_{*,t+h}(z_t)]\mathbb{E}[G_t(z_t)]dz_t}{\int_{\mathcal{W}_Z} \mathbb{E}[W'_t(z_t)]\mathbb{E}[G_t(z_t)]dz_t}$
Generalized Wald	$\frac{\mathbb{E}[Y_{t+h} \mid Z_t=z,\mathcal{F}_{t-1}]-\mathbb{E}[Y_{t+h} \mid Z_t=z',\mathcal{F}_{t-1}]}{\mathbb{E}[W_{k,t} \mid Z_t=z,\mathcal{F}_{t-1}]-\mathbb{E}[W_{k,t} \mid Z_t=z',\mathcal{F}_{t-1}]}$	$\frac{\int_{\mathcal{W}} \mathbb{E}[Y'_{t+h}(w_k) \mid H_t(w_k)=1,\mathcal{F}_{t-1}]\mathbb{E}[H_t(w_k) \mid \mathcal{F}_{t-1}]dw_k}{\int_{\mathcal{W}} \mathbb{E}[H_t(w_k) \mid \mathcal{F}_{t-1}]dw_k}$
Filtered IV	$\frac{\mathbb{E}[(Y_{t+h}-\hat{Y}_{t+h})(Z_t-\hat{Z}_t)]}{\mathbb{E}[(W_t-\hat{W}_t)(Z_t-\hat{Z}_t)]}$	$\frac{\int_{\mathcal{W}_Z} \mathbb{E}[\mathbb{E}[Y'_{*,t+h}(z_t) \mid \mathcal{F}_{t-1}]\mathbb{E}[G_t(z_t) \mid \mathcal{F}_{t-1}]]dz_t}{\int_{\mathcal{W}_Z} \mathbb{E}[\mathbb{E}[W'_t(z_t) \mid \mathcal{F}_{t-1}]\mathbb{E}[G_t(z_t) \mid \mathcal{F}_{t-1}]]dz_t}$

Table 2: Top line results for the causal interpretation of common estimands based on assignments, instruments and outcomes. Here $H_t(w_k) = 1\{W_{k,t}(z') \leq w_k \leq W_{k,t}(z)\}$, $G_t(z_t) = 1\{z_t \leq Z_t\}(Z_t - \mathbb{E}[Z_t])$ and $G_{t|t-1}(z_t) = 1\{z_t \leq Z_t\}(Z_t - \mathbb{E}[Z_t] \mid \mathcal{F}_{t-1}])$, while $\hat{Y}_{t+h|t-1} = \mathbb{E}[Y_{t+h} \mid \mathcal{F}_{t-1}]$, $\hat{Z}_t = \mathbb{E}[Z_t \mid \mathcal{F}_{t-1}]$ and $\hat{W}_t = \mathbb{E}[W_t \mid \mathcal{F}_{t-1}]$. Note that $\mathbb{E}[G_t(z_t)] \geq 0$ and $\mathbb{E}[G_{t|t-1}(z_t) \mid \mathcal{F}_{t-1}] \geq 0$.

instrument relative to the impulse response function of the assignment on the instrument. We show that such dynamic IV estimands identify local average impulse causal effects in the sense of Imbens and Angrist (1994); Angrist et al. (1996, 2000).

In particular, we ask if, for $h \geq 0$ and fixed $z, z' \in \mathcal{W}_Z$, the following statistical estimands have causal meaning: Wald, IV, generalized Wald, and filtered IV. These estimands are defined in the middle column of Table 2. The right hand column of Table 2 gives their interpretation under some important restrictions on the assignments and instruments and some additional technical conditions. The rest of this Section will spell out the details.

5.1 Wald Estimand

Under an instrumented potential outcome system the Wald estimand is

$$\frac{\mathbb{E}[Y_{t+h} \mid Z_t = z] - \mathbb{E}[Y_{t+h} \mid Z_t = z']}{\mathbb{E}[W_{k,t} \mid Z_t = z] - \mathbb{E}[W_{k,t} \mid Z_t = z']}.$$

The numerator of the Wald estimand is the impulse response of the outcome Y_{t+h} on the instrument Z_t, which can be thought of as the "reduced-form." The denominator of the the Wald estimand is the impulse response function of the assignment $W_{k,t}$ on the instrument Z_t, which can be thought of as the "first-stage." Therefore, the Wald estimand is equals a ratio of a reduced-form impulse response function to a first-stage conditional impulse response function, and therefore is a natural time series generalization of the Wald estimand from cross-sectional causal inference.

Our next result establishes that the Wald estimand identifies a weighted average of marginal causal effects for "compliers" provided that (i) the potential outcome process is continuously differentiable in the assignment; (ii) satisfies a standard monotonicity condition as introduced in Imbens and Angrist (1994); (iii) the instrument is independent of the potential assignment and outcome processes; (iv) is a type of relevance condition.

Theorem 6. *Assume an instrumented potential outcome system, fix $z, z' \in \mathcal{W}_Z$ and that*

i. *The $Y_{t+h}(w_k)$ is continuously differentiable in the closed interval $w_k \in \mathcal{W}_k := [\underline{w}_k, \overline{w}_k] \subset \mathbb{R}$.*

ii. *$W_{k,t}(z') \leq W_{k,t}(z)$ with probability one.*

iii. *The instrument satisfies*

$$[Z_t \perp\!\!\!\perp \{W_{k,t}(z) \colon z \in \mathcal{W}_Z\}], \quad [Z_t \perp\!\!\!\perp \{Y_{t+h}(w_k) \colon w_k \in \mathcal{W}_k\}]. \tag{15}$$

iv. *$\int_{\mathcal{W}} \mathbb{E}[1\{W_{k,t}(z') \leq w_k \leq W_{k,t}(z)\}] dw_k > 0$.*

Then Wald estimand equals, so long as it exists,

$$\frac{\int_{\mathcal{W}} \mathbb{E}[Y'_{t+h}(w_k) | H_t(w_k) = 1] \mathbb{E}[H_t(w_k)] dw_k}{\int_{\mathcal{W}} \mathbb{E}[H_t(w_k)] dw_k},$$

where $H_t(w_k) = 1\{W_{k,t}(z') \leq w_k \leq W_{k,t}(z)\}$.

Proof: Given in Appendix.

Assumption iii says that the instrument is randomly allocated. This implicitly restricts the relationship between:

1. the instrument Z_t and other assignments $W_{1:k-1,t:t+h}, W_{k+1:d_W,t:t+h}$;
2. the instrument Z_t and future and past potential assignments

$$\{W_{1:t-1}(z_{1:t-1}), W_{t+1:t+h}(z_{t+1:t+h}) \colon z_{1:t-1} \in \mathcal{Z}^t, z_{t+1:t+h} \in \mathcal{Z}^h\};$$

3. the instrument Z_t and its future and past values $Z_{1:t-1}$ and $Z_{t+1:t+h}$; and
4. the instrument Z_t and the potential outcome process $\{Y_{j,t+h}(w_{1:t+h}) \colon w_{1:t+h} \in \mathcal{W}^{t+h}\}$.

Remark 4 (Binary Assignment, Binary Instrument Case)**.** *Assume $W_{k,t} \in \{0, 1\}$, $Z_t \in \{0, 1\}$ and $z = 1, z' = 0$. In this case, although the math is different due to the discreteness of the assignment and instrument, under the same conditions as Theorem 8, we can show that the conditional IV estimand $IV_{k,t,h}$ becomes*

$$\mathbb{E}[\{Y_{t+h}(1) - Y_{t+h}(0)\} \mid W_{k,t}(1) - W_{k,t}(0) = 1].$$

a local average treatment effect.

5.2 IV Estimand

The instrumental regression of Y_{t+h} on W_t using the instrument Z_t is

$$IV_{k,t,h} := \frac{Cov(Y_{t+h}, Z_t)}{Cov(W_t, Z_t)}.$$

This has a causal interpretation by twice applying the local projection estimand from Theorem 3, once on the nominator for the local projection of Y_{t+h} on Z_t, once on the denominator for the local projection of W_t on Z_t.

The statement of the results uses the notation

$$Y_{*,t+h}(z_t) = Y_{t+h}(W_{1:t-1}, W_{t,1:k-1}, W_k(z_t), W_{t,k+1:d_W}, W_{t+1:t+h}),$$

and $Y'_{*,t+h}(z_t) = \partial Y_{*,t+h}(z_t)/\partial z_t$.

Theorem 7. *Assume an instrumented potential outcome system. Further assume that*

i. The support of Z_t is a closed interval $\mathcal{W}_Z : [\underline{z}, \overline{z}] \subset R$.

ii. $Y_{*,t+h}(z)$ and that $W_t(z)$ are continuously differentiable in $z \in \mathcal{W}_Z$.

iii. The

$$Z_t \perp\!\!\!\perp \{W_t(z) \colon z \in \mathcal{W}_Z\}, \quad Z_t \perp\!\!\!\perp \{Y_{*,t+h}(z) \colon z \in \mathcal{W}_Z\},$$

iv. The $\int_{\mathcal{W}_Z} \mathbb{E}[W'_t(z_t)]\mathbb{E}[G_t(z_t)]dz_t \neq 0$.

Then, it follows, if it exists, that

$$IV_{k,t,h} = \frac{\int_{\mathcal{W}_Z} \mathbb{E}[Y'_{*,t+h}(z_t)]\mathbb{E}[G_t(z_t)]dz_t}{\int_{\mathcal{W}_Z} \mathbb{E}[W'_t(z_t)]\mathbb{E}[G_t(z_t)]dz_t}$$

where $G_t(z_t) = 1\{z_t \leq Z_t\}(Z_t - \mathbb{E}[Z_t])$, noting $\mathbb{E}[G_t(z_t)] \geq 0$.

Proof. Application of Theorem 3, twice, once on the numerator and once on the demoninator. Assumption iv is a type of relevance condition.

5.3 Generalized Wald Estimand

The generalized Wald estimand is given by, for fixed $z, z' \in \mathcal{W}_Z$,

$$GW_{k,t,h} := \frac{\mathbb{E}[Y_{t+h} \mid Z_t = z, \mathcal{F}_{t-1}] - \mathbb{E}[Y_{t+h} \mid Z_t = z', \mathcal{F}_{t-1}]}{\mathbb{E}[W_{k,t} \mid Z_t = z, \mathcal{F}_{t-1}] - \mathbb{E}[W_{k,t} \mid Z_t = z', \mathcal{F}_{t-1}]}. \tag{16}$$

It equals a ratio of a reduced-form generalized impulse response function to a first-stage generalized impulse response function.

Theorem 8. *Assume an instrumented potential outcome system, fix $z, z' \in \mathcal{W}_Z$ and that*

i. The $Y_{t+h}(w_k)$ is continuously differentiable in the closed interval $w_k \in \mathcal{W}_k := [\underline{w}_k, \overline{w}_k] \subset \mathbb{R}$.

ii. $W_{k,t}(z') \leq W_{k,t}(z)$ with probability one.

iii. The instrument satisfies

$$[Z_t \perp\!\!\!\perp \{W_{k,t}(z)\colon z \in \mathcal{W}_Z\}] \mid \mathcal{F}_{t-1}, \quad [Z_t \perp\!\!\!\perp \{Y_{t+h}(w_k)\colon w_k \in \mathcal{W}_k\}] \mid \mathcal{F}_{t-1}. \tag{17}$$

iv. $\int_{\mathcal{W}} \mathbb{E}[1\{W_{k,t}(z') \leq w_k \leq W_{k,t}(z)\} \mid \mathcal{F}_{t-1}] dw_k > 0$.

For a potential outcome system with an external instrument, $GW_{k,t,h}$ equals, so long as it exists,

$$\frac{\int_{\mathcal{W}} \mathbb{E}[Y'_{t+h}(w_k) \mid H_t(w_k) = 1, \mathcal{F}_{t-1}] \mathbb{E}[H_t(w_k) \mid \mathcal{F}_{t-1}] dw_k}{\int_{\mathcal{W}} \mathbb{E}[H_t(w_k) \mid \mathcal{F}_{t-1}] dw_k},$$

where, again, $H_t(w_k) = 1\{W_{k,t}(z') \leq w_k \leq W_{k,t}(z)\}$.

Proof: Given in Appendix.

Therefore, the conditional Wald estimand is equal to a weighted average of the marginal causal effects for "compliers" (i.e., realizations of the potential treatment function for which moving the instrument from z' to z changes the assignment). The marginal causal effect is the derivative of the h-step ahead potential outcome process with respect to the k-th assignment, holding all else constant. The weights are proportional to the probability of the potential assignment function being a "complier," so are non-negative and sum to 1.

We next provide a sufficient condition for the instrument to be randomly assigned in terms of conditional independence restrictions on these underlying processes.

Theorem 9. *Assume that the instrument satisfies*

$$Z_t \perp\!\!\!\perp \big(Z_{t+1:t+h}, W_{1:k-1,t:t+h}, \{W_{k,t+1:t+h}(z_{t+1:t+h})\colon z_{t+1:t+h} \in \mathcal{Z}^h\}, W_{k+1:d_W,t:t+h},$$
$$\{Y_{t+h}(w_{1:t+h})\colon w_{1:t+h} \in \mathcal{W}^{t+h}\}\big) \mid \mathcal{F}_{t-1}.$$

Then, Assumption iii in Theorem 8 is satisfied.

Proof: Trivial.

5.4 Filtered IV Estimand

Estimating generalized Wald estimand is not easy, particularly if Z_t is not discrete. Here we derive a causal interpretation for generalized IV estimand

$$GIV_{k,t,h} := \frac{Cov(Y_{t+h}, Z_t \mid \mathcal{F}_{t-1})}{Cov(W_t, Z_t \mid \mathcal{F}_{t-1})} = \frac{\mathbb{E}[(Y_{t+h} - \hat{Y}_{t+h})(Z_t - \hat{Z}_t) \mid \mathcal{F}_{t-1}]}{\mathbb{E}[(W_t - \hat{W}_t)(Z_t - \hat{Z}_t) \mid \mathcal{F}_{t-1}]}.$$

where $\hat{Y}_{t+h} = \mathbb{E}[Y_{t+h} \mid \mathcal{F}_{t-1}]$, $\hat{W}_t = \mathbb{E}[W_t \mid \mathcal{F}_{t-1}]$ and $\hat{Z}_{t+h} = \mathbb{E}[Z_{t+h} \mid \mathcal{F}_{t-1}]$.

No new technical issue arise in dealing with this setup, but condition iii in Theorem 7 becomes

$$[Z_t \perp\!\!\!\perp \{Y_{*,t+h}(z) \colon z \in \mathcal{W}_Z\}] \mid \mathcal{F}_{t-1}, \quad [Z_t \perp\!\!\!\perp \{W_t(z) \colon z \in \mathcal{W}_Z\}] \mid \mathcal{F}_{t-1}, \tag{18}$$

which are much easier to think about. Then $GIV_{k,t,h}$ equals

$$\frac{\int_{\mathcal{W}_Z} \mathbb{E}[Y'_{*,t+h}(z_t) \mid \mathcal{F}_{t-1}]\mathbb{E}[G_{t|t-1}(z_t) \mid \mathcal{F}_{t-1}]dz_t}{\int_{\mathcal{W}_Z} \mathbb{E}[W'_t(z_t) \mid \mathcal{F}_{t-1}]\mathbb{E}[G_{t|t-1}(z_t) \mid \mathcal{F}_{t-1}]dz_t}$$

where $G_{t|t-1}(z_t) = 1\{z_t \leq Z_t\}(Z_t - \mathbb{E}[Z_t \mid \mathcal{F}_{t-1}])$, noting $\mathbb{E}[G_{t|t-1}(z_t) \mid \mathcal{F}_{t-1}] \geq 0$.

Of more practical importance is the filtered IV estimand

$$\frac{\mathbb{E}[(Y_{t+h} - \hat{Y}_{t+h})(Z_t - \hat{Z}_t)]}{\mathbb{E}[(W_t - \hat{W}_t)(Z_t - \hat{Z}_t)]},$$

which can be estimated by instrumental variables applied to $Y_{t+h} - \hat{Y}_{t+h}$ on $W_t - \hat{W}_t$ with instruments $Z_t - \hat{Z}_t$. Then, again, under the conditions of Theorem 7 but using (18) instead of condition iii, then the filtered IV estimand becomes

$$\frac{\int_{\mathcal{W}_Z} \mathbb{E}[\mathbb{E}[Y'_{*,t+h}(z_t) \mid \mathcal{F}_{t-1}]\mathbb{E}[G_{t|t-1}(z_t) \mid \mathcal{F}_{t-1}]]dz_t}{\int_{\mathcal{W}_Z} \mathbb{E}[\mathbb{E}[W'_t(z_t) \mid \mathcal{F}_{t-1}]\mathbb{E}[G_{t|t-1}(z_t) \mid \mathcal{F}_{t-1}]]dz_t}.$$

6 Estimands Based on Instruments and Outcomes

In this section, we study the nonparametric conditions under which some statistical estimands based on instruments and outcomes have causal meaning in the context of an instrumented potential outcome system

$$\{Z_t, \{W_t(z_t), z_t \in \mathcal{W}_Z\}, \{Y_t(w_{1:t}), w_{1:t} \in \mathcal{W}_W^t\}\}_{t \geq 1},$$

where researchers only see the instruments and the outcomes

$$\{z_t^{obs}, y_t^{obs}\}_{t \geq 1}.$$

In this framework we will sometimes refer to $\{\mathcal{F}_t^{Z,Y}\}_{t\geq 1}$ as the natural filtration generated by the realized $\{z_t^{obs}, y_t^{obs}\}_{t\geq 1}$, while $z, z' \in \mathcal{W}_Z$. Throughout we follow the literature and estimate terms which involve possibly two elements of the outcome vector, $Y_{j,t+h}$ and $Y_{k,t}$, and so we return to using a explicit subscript on the outcome variable.

In this context, researchers constructs a dynamic IV estimand that takes the ratio of an impulse response function of the outcome on the instrument and the impulse response function of the lagged outcome on the instrument. We show that such dynamic IV estimands identify "relative" local average impulse causal effect, which is a nonparametrically generalization of the interpretation of such a dynamic IV estimand in existing literature on external instruments (Stock and Watson, 2018; Plagborg-Møller and Wolf, 2020; Jordá et al., 2020).

As mentioned, this mirrors the typical construct of external IV estimates of impulse response functions in macroeconomics. Consider, for example, an empirical researcher that constructs an instrument Z_t for the monetary policy shock (e.g., an instrument of the form used in Kuttner (2001); Cochrane and Piazessi (2002); Faust et al. (2003); Gurkaynak et al. (2005); Bernanke and Kuttner (2005); Gertler and Karadi (2015) or Romer and Romer (2004)). In this case, the empirical researcher will estimate attempt to measure the dynamic causal effect of the monetary policy shock $W_{k,t}$ on unemployment $Y_{j,t+h}$ by estimating the first-stage impulse response function of the federal funds rate $Y_{k,t}$ on the instrument Z_t. See, for example, Jordá et al. (2015); Ramey (2016); Jordá et al. (2020); Ramey and Zubairy (2018) for recent applications.

These results underscore the value of external instruments in macroeconometric research as we show that external instruments can be used to identify nonparametric causal estimands without functional form assumptions and without observing the assignment.

In particular, we ask if, for $h \geq 0$ and fixed $z, z' \in \mathcal{W}_Z$, the following statistical estimands have causal meaning: Ratio Wald, Local Projection IV, generalized Ratio Wald, and the local filtered projection IV. These estimands are defined in the middle column of Table 3. The top line results are they have the interpretation given in the right hand column of Table 3 under some important restrictions on the assignments and instruments and some additional technical conditions. The rest of this Section will spell out the details.

6.1 Ratio Wald Estimand

The Ratio Wald Estimand

$$\frac{\mathbb{E}[Y_{j,t+h} \mid Z_t = z] - \mathbb{E}[Y_{j,t+h} \mid Z_t = z']}{\mathbb{E}[Y_{k,t} \mid Z_t = z] - \mathbb{E}[Y_{k,t} \mid Z_t = z']},$$

Name	Estimand	Causal Interpretation
Ratio Wald	$\frac{\mathbb{E}[Y_{j,t+h}\|Z_t=z]-\mathbb{E}[Y_{j,t+h}\|Z_t=z']}{\mathbb{E}[Y_{k,t}\|Z_t=z]-\mathbb{E}[Y_{k,t}\|Z_t=z']}$	$\frac{\int_{\mathcal{W}}\mathbb{E}[Y'_{j,t+h}(w_k)\|H_t(w_k)=1]\mathbb{E}[H_t(w_k)]dw_k}{\int_{\mathcal{W}}\mathbb{E}[Y'_{k,t}(w_k)\|H_t(w_k)=1]\mathbb{E}[H_t(w_k)]dw_k}$
Local Projection IV	$\frac{Cov(Y_{j,t+h},Z_t)}{Cov(Y_{k,t},Z_t)}$	$\frac{\int_{\mathcal{W}_Z}\mathbb{E}[Y'_{j,t+h}(z_k)]\mathbb{E}[G_t(z_k)]dz_k}{\int_{\mathcal{W}_Z}\mathbb{E}[Y'_{k,t}(z_k)]\mathbb{E}[G_t(z_k)]dz_k}$
Generalized Ratio Wald	$\frac{\mathbb{E}[Y_{j,t+h}\|Z_t=z,\mathcal{F}^{Z,Y}_{t-1}]-\mathbb{E}[Y_{j,t+h}\|Z_t=z',\mathcal{F}^{Z,Y}_{t-1}]}{\mathbb{E}[Y_{k,t}\|Z_t=z,\mathcal{F}^{Z,Y}_{t-1}]-\mathbb{E}[Y_{k,t}\|Z_t=z',\mathcal{F}^{Z,Y}_{t-1}]}$	$\frac{\int_{\mathcal{W}}\mathbb{E}[Y'_{j,t+h}(w_k)\|H_t(w_k)=1,\mathcal{F}^{Z,Y}_{t-1}]\mathbb{E}[H_t(w_k)\|\mathcal{F}^{Z,Y}_{t-1}]dw_k}{\int_{\mathcal{W}}\mathbb{E}[Y'_{k,t}(w_k)\|H_t(w_k)=1,\mathcal{F}^{Z,Y}_{t-1}]\mathbb{E}[H_t(w_k)\|\mathcal{F}^{Z,Y}_{t-1}]dw_k}$
Local Filtered Projection IV	$\frac{Cov(Y_{j,t+h}-\hat{Y}_{j,t+h},Z_t-\hat{Z}_t)}{Cov(Y_{j,t}-\hat{Y}_{k,t},Z_t-\hat{Z}_t)}$	$\frac{\int_{\mathcal{W}_Z}\mathbb{E}[\mathbb{E}[Y'_{j,t+h}(z_k)\|\mathcal{F}^{Z,Y}_{t-1}]\mathbb{E}[G_t(z_k)\|\mathcal{F}^{Z,Y}_{t-1}]]dz_k}{\int_{\mathcal{W}_Z}\mathbb{E}[\mathbb{E}[Y'_{k,t}(z_k)\|\mathcal{F}^{Z,Y}_{t-1}]\mathbb{E}[G_t(z_k)\|\mathcal{F}^{Z,Y}_{t-1}]]dz_k}$

Table 3: Top line results for the causal interpretation of common estimands based on instruments and outcomes. Here $H_t(w_k) = 1\{W_{k,t}(z') \leq w_k \leq W_{k,t}(z)\}$, $G_t(z_t) = 1\{z_t \leq Z_t\}(Z_t - \mathbb{E}[Z_t])$ and $G_{t|t-1}(z_t) = 1\{z_t \leq Z_t\}(Z_t - \mathbb{E}[Z_t \mid \mathcal{F}^{Z,Y}_{t-1}])$, while $\hat{Y}_{t+h|t-1} = \mathbb{E}[Y_{t+h} \mid \mathcal{F}^{Z,Y}_{t-1}]$ and $\hat{Z}_t = \mathbb{E}[Z_t \mid \mathcal{F}^{Z,Y}_{t-1}]$. Note that $\mathbb{E}[G_t(z_t)] \geq 0$ and $\mathbb{E}[G_{t|t-1}(z_t) \mid \mathcal{F}^{Z,Y}_{t-1}] \geq 0$.

which is the ratio of the Wald estimands:

$$\frac{\mathbb{E}[Y_{j,t+h} \mid Z_t = z] - \mathbb{E}[Y_{j,t+h} \mid Z_t = z']}{\mathbb{E}[W_{k,t} \mid Z_t = z] - \mathbb{E}[W_{k,t} \mid Z_t = z']}, \quad \text{to} \quad \frac{\mathbb{E}[Y_{k,t} \mid Z_t = z] - \mathbb{E}[Y_{k,t} \mid Z_t = z']}{\mathbb{E}[W_{k,t} \mid Z_t = z] - \mathbb{E}[W_{k,t} \mid Z_t = z']}.$$

Hence we just need to collect the conditions for the validity of their causal representations. This is carried out in the following Corollary.

Corollary 1. *Assume an instrumented potential outcome system, $z, z' \in \mathcal{W}_Z$ and that*

i. $Y_{k,t}(w_k), Y_{j,t+h}(w_k)$ *are continuously differentiable in closed interval* $\mathcal{W}_k := [\underline{w}_k, \overline{w}_k] \subset \mathbb{R}$.

ii. $W_{k,t}(z') \leq W_{k,t}(z)$ *with probability one.*

iii. The instrument satisfies

$$[Z_t \perp\!\!\!\perp \{W_{k,t}(z) \colon z \in \mathcal{W}_Z\}], \quad [Z_t \perp\!\!\!\perp \{Y_{k,t}(w_k), Y_{j,t+h}(w_k) \colon w_k \in \mathcal{W}_k\}]. \tag{19}$$

iv. $\int_{\mathcal{W}} \mathbb{E}[Y'_{k,t}(w_k) \mid H_t(w_k) = 1]\mathbb{E}[H_t(w_k)]dw_k \neq 0$.

Then the Ratio Wald Estimand equals, if it exists and $H_t(w_k) = 1\{W_{k,t}(z') \leq w_k \leq W_{k,t}(z)\}$,

$$\frac{\int_{\mathcal{W}} \mathbb{E}[Y'_{k,t+h}(w_k) \mid H_t(w_k) = 1]\mathbb{E}[H_t(w_k)]dw_k}{\int_{\mathcal{W}} \mathbb{E}[Y'_{k,t}(w_k) \mid H_t(w_k) = 1]\mathbb{E}[H_t(w_k)]dw_k}.$$

Proof. It is implied by Theorem 6, used twice.

6.2 Local Projection IV Estimand

The local projection IV estimand

$$\frac{Cov(Y_{j,t+h}, Z_t)}{Cov(Y_{k,t}, Z_t)},$$

is the ratio of the IV estimands

$$\frac{Cov(Y_{j,t+h}, Z_t)}{Cov(W_t, Z_t)}, \quad \text{to} \quad \frac{Cov(Y_{k,t}, Z_t)}{Cov(W_t, Z_t)}.$$

Hence we just need to collect the conditions for the validity of their causal representations. This is carried out in the following Corollary.

Corollary 2. *Assume an instrumented potential outcome system. Further assume that*

i. $\mathcal{W}_Z : [\underline{z}, \overline{z}] \subset R$*, a closed interval.*

ii. $Y_{j,t}(z), Y_{k,t+h}(z), W_t(z)$ *are continuously differentiable in* $z \in \mathcal{W}_Z$.

iii. The

$$Z_t \perp\!\!\!\perp \{Y_{k,t}(z), Y_{j,t+h}(z) \colon z \in \mathcal{W}_Z\}, \quad Z_t \perp\!\!\!\perp \{W_t(z) \colon z \in \mathcal{W}_Z\}.$$

iv. The $\int_{\mathcal{W}_Z} \mathbb{E}[Y'_{k,t}(z_t)]\mathbb{E}[G_t(z_t)]dz_t \neq 0$.

Then the local projection IV estimand equals

$$\frac{\int_{\mathcal{W}_Z} \mathbb{E}[Y'_{j,t+h}(z_k)]\mathbb{E}[G_t(z_k)]dz_k}{\int_{\mathcal{W}_Z} \mathbb{E}[Y'_{k,t}(z_k)]\mathbb{E}[G_t(z_k)]dz_k},$$

where $G_t(z_k) = 1\{z_k \leq Z_t\}(Z_t - \mathbb{E}[Z_t])$*, noting* $\mathbb{E}[G_t(z_k)] \geq 0$.

Proof. It is implied by Theorem 7, used twice.

6.3 Generalized Ratio Wald Estimand

It is common for researchers to estimate the generalized ratio Wald estimand:

$$\frac{\mathbb{E}[Y_{j,t+h} \mid Z_t = z, \mathcal{F}^{Z,Y}_{t-1}] - \mathbb{E}[Y_{j,t+h} \mid Z_t = z', \mathcal{F}^{Z,Y}_{t-1}]}{\mathbb{E}[Y_{k,t} \mid Z_t = z, \mathcal{F}^{Z,Y}_{t-1}] - \mathbb{E}[Y_{k,t} \mid Z_t = z', \mathcal{F}^{Z,Y}_{t-1}]},$$

the ratio of generalized impulse response functions at different lags, for different outcome variables. But this is the ratio of two generalized Wald estimands.

Corollary 3. *Assume an instrumented potential outcome system,* $z, z' \in \mathcal{W}_Z$ *and that* *i.* $Y_{k,t}(w_k), Y_{j,t+h}(w_k)$ *are continuously differentiable in closed interval* $\mathcal{W}_k := [\underline{w}_k, \overline{w}_k] \subset \mathbb{R}$.

ii. $W_{k,t}(z') \leq W_{k,t}(z) \mid \mathcal{F}_{t-1}^{Z,Y}$ *with probability one.*

iii. The instrument satisfies

$$[Z_t \perp\!\!\!\perp \{W_{k,t}(z)\colon z \in \mathcal{W}_Z\}] \mid \mathcal{F}_{t-1}^{Z,Y}, \quad [Z_t \perp\!\!\!\perp \{Y_{k,t}(w_k), Y_{j,t+h}(w_k)\colon w_k \in \mathcal{W}_k\}] \mid \mathcal{F}_{t-1}^{Z,Y}.$$

iv. $\int_{\mathcal{W}} \mathbb{E}[Y'_{k,t}(w_k) \mid H_t(w_k) = 1, \mathcal{F}_{t-1}^{Z,Y}]\mathbb{E}[H_t(w_k) \mid \mathcal{F}_{t-1}^{Z,Y}]dw_k \neq 0.$

Then, generalized ratio Wald estimand equals, where $H_t(w_k) = 1\{W_{k,t}(z') \leq w_k \leq W_{k,t}(z)\}$,

$$\frac{\int_{\mathcal{W}} \mathbb{E}[Y'_{k,t+h}(w_k) \mid H_t(w_k) = 1, \mathcal{F}_{t-1}^{Z,Y}]\mathbb{E}[H_t(w_k) \mid \mathcal{F}_{t-1}^{Z,Y}]dw_k}{\int_{\mathcal{W}} \mathbb{E}[Y'_{k,t}(w_k) \mid H_t(w_k) = 1, \mathcal{F}_{t-1}^{Z,Y}]\mathbb{E}[H_t(w_k) \mid \mathcal{F}_{t-1}^{Z,Y}]dw_k}.$$

Proof. It is implied by Theorem 8, used twice.

In words, the conditional IV estimand above identifies a *relative* local average impulse causal under the potential outcome system. The numerator and the denominator have the same interpretation as Theorem 8. In particular, the numerator is a weighted average of the marginal causal effects of $W_{k,t}$ on the h-step ahead outcome $Y_{j,t+h}$, where the weights are proportional to the probability of compliance. Similarly, the denominator is a weighted average of the marginal causal effects of $W_{k,t}$ on the contemporaneous outcome $Y_{k,t}$. Therefore, we can interpret their ratio as measuring the causal response of the h-step ahead outcome $Y_{j,t+h}$ to a change in the treatment $W_{k,t}$ that increases the contemporaneous outcome $Y_{k,t}$ by one unit on impact (among compliers).

This is a nonparametric generalization of the result that in linear SVMA models (without invertibility) the IV based estimands identify relative impulse response functions (Stock and Watson (2018); Plagborg-Møller and Wolf (2020)). This result makes no functional form assumptions nor standard time series assumptions such as invertibility or recoverability. In this sense, Corollary 3 highlights the attractiveness of using external instruments to measure dynamic causal effects in observational time series data. Provided there exists an external instrument for the treatment $W_{k,t}$, then the researcher can robustly identify causally interpretable estimands without further assumptions and without even directly observing the treatment itself.

6.4 Local Filtered Projection IV Estimand

In practice researchers typically estimate generalized impulse response functions using a two-stage least-squares type estimator. This is also sometimes called "local projections with an external instrument" (Jordá et al., 2015). We first analyze this generalized local projection IV

$$\frac{Cov(Y_{j,t+h}, Z_t \mid \mathcal{F}_{t-1}^{Z,Y})}{Cov(Y_{k,t}, Z_t \mid \mathcal{F}_{t-1}^{Z,Y})}, \tag{20}$$

which again is a ratio, this time of the Generalized IV estimands at different lag lengths. Using the same arguments as Corollary 2, it has the causal interpretation

$$\frac{\int_{\mathcal{W}_Z} \mathbb{E}[Y'_{j,t+h}(z_k) \mid \mathcal{F}_{t-1}^{Z,Y}]\mathbb{E}[G_t(z_k) \mid \mathcal{F}_{t-1}^{Z,Y}]dz_k}{\int_{\mathcal{W}_Z} \mathbb{E}[Y'_{k,t}(z_k) \mid \mathcal{F}_{t-1}^{Z,Y}]\mathbb{E}[G_t(z_k) \mid \mathcal{F}_{t-1}^{Z,Y}]dz_k},$$

where $G_{t|t-1}(z_k) = 1\{z_k \leq Z_t\}(Z_t - \mathbb{E}[Z_t \mid \mathcal{F}_{t-1}^{Z,Y}])$.

Of more practical relevance, is the local filtered projection IV estimand is

$$\frac{Cov(Y_{j,t+h} - \hat{Y}_{j,t+h}, Z_t - \hat{Z}_t)}{Cov(Y_{k,t} - \hat{Y}_{k,t}, Z_t - \hat{Z}_t)},$$

where $\hat{Y}_{t+h} = \mathbb{E}[Y_{t+h} \mid \mathcal{F}_{t-1}^{Z,Y}]$, $\hat{Y}_t = \mathbb{E}[Y_t \mid \mathcal{F}_{t-1}^{Z,Y}]$ and $\hat{Z}_{t+h} = \mathbb{E}[Z_{t+h} \mid \mathcal{F}_{t-1}^{Z,Y}]$. The properties of this are inherited from those of the generalized local projection IV. In particular it equals

$$\frac{\int_{\mathcal{W}_Z} \mathbb{E}[\mathbb{E}[Y'_{j,t+h}(z_k) \mid \mathcal{F}_{t-1}^{Z,Y}]\mathbb{E}[G_t(z_k) \mid \mathcal{F}_{t-1}^{Z,Y}]]dz_k}{\int_{\mathcal{W}_Z} \mathbb{E}[\mathbb{E}[Y'_{k,t}(z_k) \mid \mathcal{F}_{t-1}^{Z,Y}]\mathbb{E}[G_t(z_k) \mid \mathcal{F}_{t-1}^{Z,Y}]]dz_k}.$$

7 Estimands Based Only on Outcomes

The dominant approach to causal inference in macroeconometrics is a model-based approach in the tradition of Sims (1980). In that literature, researchers introduce parametric models as they wish to study the dynamic causal effects of unobservable "structural shocks," which themselves must be inferred from the outcomes. Here we link this to our setup, mostly to place our work in context. Assume there is a potential outcome system

$$\{W_t, \{Y_t(w_{1:t}) : w_{1:t} \in \mathcal{W}_w^t\}_{t\geq 1},$$

where researchers only see the outcomes

$$\{y_t^{obs}\}_{t\geq 1}.$$

7.1 Linear simultaneous equation approach

The causal inference approach of using only time series data on outcomes is in the storied tradition of linear simultaneous equations models developed at the Cowles Foundation (e.g. Christ (1994), Hausman (1983)). The time series aspects are mostly technical, the most essential causal ideas appear in the cross-section, so we start there.

The approach is model-based, here we focus on the linear case

$$A_0 Y_t(w_{1:t}) = \alpha + w_t, \quad w_{1:t} \in \mathcal{W}^t, \quad t = 1, 2, ...,$$

where A_0 is a non-stochastic, square matrix. Notice that in this model the potential outcomes are not stochastic — this linear case says that linear combinations of the potential outcomes equal the possible assignments for every t.

Now additionally assume that A_0 is invertible, then

$$Y_t(w_{1:t}) = A_0^{-1}(\alpha + w_t), \quad \frac{\partial Y_t(w_{1:t})}{\partial w_t^{\mathrm{T}}} = A_0^{-1},$$

and the contemporaneous average treatment effect

$$\begin{aligned} \mathrm{E}[Y_t(W_{1:t-1}, w) - Y_t(W_{1:t-1}, w')] &= Y_t(W_{1:t-1}, w) - Y_t(W_{1:t-1}, w') \\ &= A_0^{-1}(w - w'), \end{aligned}$$

whatever probabilistic assumption is made about $W_{1:t-1}$. Under this model, if we see $(W_t, Y_t) = \{W_t, Y_t(W_{1:t})\}$, then, if the second moments of the observables exist and $\mathrm{Var}(W_t)$ is non-singular, then for every t,

$$\mathrm{Cov}(Y_t, W_t)\mathrm{Var}(W_t)^{-1} = A_0^{-1},$$

which would make statistical inference rather straightforward. But the point of this simultaneous equations literature is to carry out inference without observing the assignments — which is a much harder task.

If, in addition to A_0 being invertible, we assume that $\mathrm{Var}(W_t) < \infty$, then

$$\mathrm{Var}(Y_t) = A_0^{-1}\mathrm{Var}(W_t)\left(A_0^{-1}\right)^{\mathrm{T}},$$

Crucially knowing $\mathrm{Var}(Y_t)$ is not enough to untangle A_0 and $\mathrm{Var}(W_t)$, and so is not enough alone to learn the contemporaneous average treatment effect. In the simultaneous equations literature this is resolved by a priori imposing more structure on the problem. This can be carried out in many different ways, inspired by the problem at hand.

A central a priori constraint is the one highlighted by Sims (1980). He imposed that (a) A_0 is triangular, (b) $\mathrm{Var}(W_t)$ is diagonal. For simplicity of exposition, look at the two dimensional case and write

$$A_0 = \begin{pmatrix} 1 & 0 \\ -a_{21} & 1 \end{pmatrix}, \quad A_0^{-1} = \begin{pmatrix} 1 & 0 \\ a_{21} & 1 \end{pmatrix}, \quad \mathrm{Var}(W_t) = \begin{pmatrix} \sigma_{11}^2 & 0 \\ 0 & \sigma_{22}^2 \end{pmatrix},$$

then the elements within A_0 and $\mathrm{Var}(W_t)$ can be individually determined from $\mathrm{Var}(Y_t)$ if $\mathrm{Var}(Y_t)$ is of full rank. The same holds in higher dimensions. Hence under a priori restrictions, the contemporaneous causal effect can be determined from the data on the outcomes, without having seen the assignments (or without the access to instruments). There are alternative a priori constraints to this triangular which also work here and the above structure extends to non-linear systems of equations $g\left\{Y_t(w_{1:t})\right\} = w_t$.

The "structural vector autoregressive" (SVAR) version of the linear simultaneous equation has the same fundamental structure. focusing on the one lag model with no intercept for simplicity, the

$$A_0 Y_t(w_{1:t}) = w_t + A_1 Y_{t-1}(w_{1:t-1}).$$

Kilian and Lutkepohl (2017) provide a book length review of this model structure and its various extensions and implications.

Then $A_0\left(I - \Phi_1 L\right) Y_t(w_{1:t}) = w_t$, where L is a lag operator and $\Phi_1 = A_0^{-1} A_1$. So

$$Y_t(w_{1:t}) = A_0^{-1} w_t + \Phi_1 Y_{t-1}(w_{1:t-1}),$$

which implies that

$$Y_t(w_{1:t}) = A_0^{-1} w_t + \Phi_1 A_0^{-1} w_{t-1} + \Phi_1^2 A_0^{-1} w_{t-2} + ... + \Phi_1^{t-1} A_0^{-1} w_1 + \Phi_1^t A_0^{-1} Y_0$$

a SVMA model. Then

$$\partial Y_{t+h}(w_{1:t+h}) / \partial w_t' = \Phi_1^h A_0^{-1},$$

and the impulse response function

$$\mathrm{E}[Y_{t+h}(W_{1:t-1}, w, W_{t+1:t+h}) - Y_{t+h}(W_{1:t-1}, w', W_{t+1:t+h})] = \Phi_1^h A_0^{-1}\left(w - w'\right).$$

The time series parameter Φ_1 can be determined from the dynamics of the data if this process is stationary. But again A_0 and $\mathrm{Var}(W_t)$ cannot be seperately pulled apart from the data, so a priori model assumptions are needed. The triangular assumption on A_0, pioneered by Sims (1980), is popular for SVAR models. Imposing that assumption takes the causal problem back to a standard parametric statistical inference problem of working with vector autoregressions.

7.2 Causal meaning of the GIRF of $Y_{k,t}$ on $Y_{j,t+h}$

A broader analysis focuses on the h-step ahead generalized impulse response function of the k-th outcome on the j-th outcome. Here we provide a nonparametric causal meaning to it in terms of potential outcomes.

Assumption 5. *Assume a potential outcome system. If $Y_t(w_{1:t})$ is deterministic for all $w_{1:t} \in \mathcal{W}^t$, then this is called a deterministic potential outcome system.*

Assumption 6. *For a deterministic potential outcome system, additionally assume that (a) for all $t \neq s$, the $W_t \perp\!\!\!\perp W_s$; (b) that $W_{k,t} \perp\!\!\!\perp W_{j,t}$ for all $j \neq k$.*

Theorem 10 quantifies what causally happens to the conditional expectations of potential outcomes when the conditional distribution of the entire $W_{1:t}$ shifts due to moves in conditioning on $Y_{k,t}$. This has some intellectual connections to the non-GIRF stochastic intervention work of, for example, Stock (1989); Munoz and van der Laan (2012); Wu et al. (2021); Papadogeorgou et al. (2019)).

Theorem 10. *Assume a deterministic potential outcome system and that Assumption 6(a) holds. Then, so long as the corresponding moments exist,*

$$\mathrm{E}[Y_{j,t+h}|(Y_{k,t}=y_k),\mathcal{F}^Y_{t-1}] - \mathrm{E}[Y_{j,t+h}|(Y_{k,t}=y'_k),\mathcal{F}^Y_{t-1}] \tag{21}$$

$$= \mathrm{E}[\psi_{j,t+h}(W_{1:t})|(Y_{k,t}=y_k),\mathcal{F}^Y_{t-1}] - \mathrm{E}[\psi_{j,t+h}(W_{1:t})|(Y_{k,t}=y'_k),\mathcal{F}^Y_{t-1}], \tag{22}$$

where $\psi_{j,t+h}(w_{1:t}) := \mathrm{E}[Y_{j,t+h}(w_{1:t}, W_{t+1:t+h})]$.

Proof. Given in the Appendix.

Overall this is quite a complicated causal effect, as it allows all the assignments from time 1 to time t to move.

8 Conclusion

This paper sets out a nonparametric foundation for making causal statements from time series of outcomes, assignments and instruments. The potential outcome system, and the instruments version, sets out how outcomes move with assignments.

The foundation is used to derive conditions under which common statistical estimands, such as impulse response functions and local projection, have causal meaning.

The paper split this analysis into three sets. The first was where outcomes and assignments are seen. The second is where outcomes, assignments and instruments are observed. The final sets is where only outcomes and instruments are observed.

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A Proofs of Results for Assignments and Outputs

A.1 Proof of Theorem 1

To prove this result, we begin by rewriting $\mathbb{E}[Y_{j,t+h}1\{W_{k,t}=w_k\}]$. Notice that

$$\mathbb{E}[Y_{j,t+h}1\{W_{k,t}=w_k\}]$$

$$=\mathbb{E}[Y_{j,t+h}(W_{1:t-1},w_k,W_{-k,t},W_{t+1:t+h})1\{W_{k,t}=w_k\}]$$

$$=\mathbb{E}[Y_{j,t+h}(W_{1:t-1},w_k,W_{-k,t},W_{t+1:t+h})]\mathbb{E}[1\{W_{k,t}=w_k\}]$$

$$+Cov\left(Y_{j,t+h}(W_{1:t-1},w_k,W_{-k,t},W_{t+1:t+h}),1\{W_{k,t}=w_k\}\right).$$

Therefore, it immediately follows that

$$\mathbb{E}[Y_{j,t+h}\mid W_{k,t}=w_k]=\mathbb{E}[Y_{j,t+h}(W_{1:t-1},w_k,W_{-k,t},W_{t+1:t+h})]$$

$$+\frac{Cov\left(Y_{j,t+h}(W_{1:t-1}, w_k, W_{-k,t}, W_{t+1:t+h}), 1\{W_{k,t} = w_k\}\right)}{\mathbb{E}[1\{W_{k,t} = w_k\}]}.$$

The result is then immediate by (i) applying the same calculation to $\mathbb{E}[Y_{j,t+h}1\{W_{k,t} = w_k'\}]$, (ii) taking the difference, and (iii) applying the definition of $Y_{j,t+h}(w_k)$. □

A.2 Proof of Theorem 3

The style proof extends Angrist et al. (2000) in their analysis of the cross-sectional Wald estimator in the cross-section. Begin by writing $Y_{t+h} = Y_{t+h}(W_{k,t})$ as

$$\begin{aligned} Y_{t+h} &= Y_{t+h}(\underline{w}_k) + \int_{\underline{w}_k}^{W_{k,t}} \frac{\partial Y_{t+h}(\tilde{w}_k)}{d\tilde{w}_k} \partial\tilde{w}_k \\ &= Y_{t+h}(\underline{w}_k) + \int_{\underline{w}_k}^{\overline{w}_k} \frac{\partial Y_{t+h}(\tilde{w}_k)}{\partial\tilde{w}_k} 1\{\tilde{w}_k \leq W_{k,t}\} d\tilde{w}_k \end{aligned}$$

by the fundamental theorem of calculus. Then, it follows that

$$\begin{aligned} &Cov(Y_{t+h}, W_{k,t}) = \mathbb{E}[Y_{t+h}(W_{k,t} - \mathbb{E}[W_{k,t}])] \\ &\stackrel{(1)}{=} \mathbb{E}[(Y_{t+h} - Y_{t+h}(\underline{w}_k))(W_{k,t} - \mathbb{E}[W_{k,t}])] \\ &= \mathbb{E}\left[\left(\int_{\underline{w}_k}^{\overline{w}_k} \frac{\partial Y_{t+h}(\tilde{w}_k)}{\partial\tilde{w}_k} 1\{\tilde{w}_k \leq W_{k,t}\} d\tilde{w}_k\right)(W_{k,t} - \mathbb{E}[W_{k,t}])\right] \\ &= \int_{\underline{w}_k}^{\overline{w}_k} \mathbb{E}\left[\frac{\partial Y_{t+h}(\tilde{w}_k)}{\partial\tilde{w}_k} 1\{\tilde{w}_k \leq W_{k,t}\}(W_{k,t} - \mathbb{E}[W_{k,t}])\right] d\tilde{w}_k \\ &\stackrel{(2)}{=} \int_{\underline{w}_k}^{\overline{w}_k} \mathbb{E}\left[\frac{\partial Y_{t+h}(\tilde{w}_k)}{\partial\tilde{w}_k}\right] \mathbb{E}\left[1\{\tilde{w}_k \leq W_{k,t}\}(W_{k,t} - \mathbb{E}[W_{k,t}])\right] d\tilde{w}_k \end{aligned}$$

where (1) and (2) follow since $W_{k,t} \perp\!\!\!\perp \{Y_{t+h}(w_k) \colon w_k \in \mathcal{W}_k\}$. Interchanging the order of the derivation and the expectation delivers the result. Analogously,

$$W_{k,t} = \underline{w}_k + \int_{\underline{w}_k}^{W_{k,t}} d\tilde{w}_k = \underline{w}_k + \int_{\underline{w}_k}^{\overline{w}_k} 1\{\tilde{w}_k \leq W_{k,t}\} d\tilde{w}_k,$$

so

$$Var(W_{k,t}) = \mathbb{E}[(W_{k,t} - \underline{w}_k)(W_{k,t} - \mathbb{E}[W_{k,t}])] = \int_{\underline{w}_k}^{\overline{w}_k} \mathbb{E}\left[1\{\tilde{w}_k \leq W_{k,t}\}(W_{k,t} - \mathbb{E}[W_{k,t}])\right] d\tilde{w}_k.$$

The result then follows immediately. To see that the resulting weights are non-negative, observe that for $\tilde{w}_k \in [\underline{w}_k, \overline{w}_k]$

$$\mathbb{E}\left[1\{W_{k,t} \geq \tilde{w}_k\}\left(W_{k,t} - \mathbb{E}[W_{k,t}]\right)\right]$$

$$= \mathbb{E}\left[1\{W_{k,t} \geq \tilde{w}_k\}W_{k,t}\right] - \mathbb{E}[1\{W_{k,t} \geq \tilde{w}_k\}]\mathbb{E}[W_{k,t}]$$

$$= \left(\mathbb{E}\left[W_{k,t} \mid W_{k,t} \geq \tilde{w}_k\right] - \mathbb{E}[W_{k,t}]\right)\mathbb{P}(W_{k,t} \geq \tilde{w}_k) \geq 0$$

since $\mathbb{E}\left[W_{k,t} \mid W_{k,t} \geq \tilde{w}_k\right] \geq \mathbb{E}[W_{k,t}]$ for $\tilde{w}_k \in [\underline{w}_k, \overline{w}_k]$. □

A.3 Proof of Theorem 4

The proof is analogous to the proof of Theorem 1. We start by rewriting $\mathbb{E}[Y_{j,t+h}1\{W_{k,t} = w_k\} \mid \mathcal{F}_{t-1}]$, noticing that

$$\mathbb{E}[Y_{j,t+h}1\{W_{k,t} = w_k\} \mid \mathcal{F}_{t-1}]$$

$$= \mathbb{E}[Y_{j,t+h}(w^{obs}_{1:t-1}, w_k, W_{-k,t}, W_{t+1:t+h})1\{W_{k,t} = w_k\} \mid \mathcal{F}_{t-1}]$$

$$= \mathbb{E}[Y_{j,t+h}(w^{obs}_{1:t-1}, w_k, W_{-k,t}, W_{t+1:t+h}) \mid \mathcal{F}_{t-1}]\mathbb{E}[1\{W_{k,t} = w_k\} \mid \mathcal{F}_{t-1}]$$

$$+ Cov\left(Y_{j,t+h}(w^{obs}_{1:t-1}, w_k, W_{-k,t}, W_{t+1:t+h}), 1\{W_{k,t} = w_k\} \mid \mathcal{F}_{t-1}\right).$$

Therefore, we have shown that

$$\mathbb{E}[Y_{j,t+h} \mid W_{k,t} = w_k, \mathcal{F}_{t-1}] = \mathbb{E}[Y_{j,t+h}(w^{obs}_{1:t-1}, w_k, W_{-k,t}, W_{t+1:t+h}) \mid \mathcal{F}_{t-1}]$$

$$+ \frac{Cov\left(Y_{j,t+h}(w^{obs}_{1:t-1}, w_k, W_{-k,t}, W_{t+1:t+h}), 1\{W_{k,t} = w_k\} \mid \mathcal{F}_{t-1}\right)}{\mathbb{E}[1\{W_{k,t} = w_k\} \mid \mathcal{F}_{t-1}]}.$$

The result follows by (i) applying the same calculation to $\mathbb{E}[Y_{j,t+h}1\{W_{k,t} = w'_k\} \mid \mathcal{F}_{t-1}]$, (ii) taking the difference, and (iii) applying the definition of the potential outcome $Y_{j,t+h}(w_k)$. □

B Proofs of Results for Assignments, Instruments and Outputs

B.1 Proof of Theorem 6

To prove this result, we first observe that

$$\mathbb{E}[Y_{j,t+h} \mid Z_t = z] = \mathbb{E}[Y_{j,t+h}(w^{obs}_{1:t-1}, W_{k,t}(z), W_{-k,t}, W_{t+1:t+h}) \mid Z_t = z]$$

$$= \mathbb{E}[Y_{j,t+h}(w^{obs}_{1:t-1}, W_{k,t}(z), W_{-k,t}, W_{t+1:t+h})]$$

by (iii). Therefore,

$$\mathbb{E}[Y_{j,t+h} \mid Z_t = z] - \mathbb{E}[Y_{j,t+h} \mid Z_t = z']$$
$$= \mathbb{E}[Y_{j,t+h}(w^{obs}_{1:t-1}, W_{k,t}(z), W_{-k,t}, W_{t+1:t+h}) - Y_{j,t+h}(w^{obs}_{1:t-1}, W_{k,t}(z'), W_{-k,t}, W_{t+1:t+h})].$$

Next, we can further rewrite this previous expression as

$$\mathbb{E}[\int_{W_{j,t}(z')}^{W_{j,t}(z)} \frac{\partial Y_{j,t+h}(w_k)}{\partial w_k} dw_k] = \mathbb{E}[\int_{\mathcal{W}} \frac{\partial Y_{j,t+h}(w_k)}{\partial w_k} 1\{W_{k,t}(z') \leq w_k \leq W_{k,t}(z)\} dw_k]$$

where we used the definition $Y_{j,t+h}(w_k) := Y_{j,t+h}(W_{1:t-1}, w_{k,t}, W_{-k,t}, W_{t+1:t+h})$. Finally, assuming that we can exchange the order of integration and expectation, we arrive at

$$\int_{\mathcal{W}} \mathbb{E}[\frac{\partial Y_{j,t+h}(w_k)}{\partial w_k} 1\{W_{k,t}(z') \leq w_k \leq W_{k,t}(z)\}] dw_k$$
$$= \int_{\mathcal{W}} \mathbb{E}[\frac{\partial Y_{j,t+h}(w_k)}{\partial w_k}, W_{k,t}(0) \leq w_k \leq W_{k,t}(1)] \mathbb{E}[1\{W_{k,t}(z') \leq w_k \leq W_{k,t}(z)\}] dw_k.$$

We may apply the same argument to the denominator (again assuming that we can exchange the order of integration and expectation) to arrive at

$$\mathbb{E}[W_{k,t} \mid Z_t = z] - \mathbb{E}[W_{k,t} \mid Z_t = z'] =$$

$$\mathbb{E}[W_{k,t}(z) - W_{k,t}(z')] = \int_{\mathcal{W}} \mathbb{E}[1\{W_{k,t}(z') \leq w_k \leq W_{k,t}(z)\}].$$

Taking the ratio then delivers the desired result. □

B.2 Proof of Theorem 8

The proof is the same as the Proof of Theorem 6, except we must now condition on $\mathcal{F}_{t-1}$ throughout. □

C Proofs of Results for Outputs

C.1 Proof of Theorem 10

Then, if the subsequent moments exist, we have that

$$\begin{aligned}
\mathrm{E}[Y_{j,t+h}|(Y_{k,t}=y_k),\mathcal{F}^Y_{t-1}] &= \mathrm{E}[Y_{j,t+h}(W_{1:t})|(Y_{k,t}=y_k),\mathcal{F}^Y_{t-1}], \quad \text{Assumption 5} \\
&= \mathrm{E}[\mathrm{E}[Y_{j,t+h}(W_{1:t+h})|(Y_{k,t}=y_k),W_{1:t},\mathcal{F}^Y_{t-1}]|Y_{k,t}=y_k,\mathcal{F}^Y_{t-1}], \quad \text{Adam's law} \\
&= \mathrm{E}[\mathrm{E}[Y_{j,t+h}(W_{1:t+h})|W_{1:t})]|(Y_{k,t}=y_k),\mathcal{F}^Y_{t-1}], \quad \text{Assumption 5} \\
&= \mathrm{E}[\psi_{j,t+h}(W_{1:t})|(Y_{k,t}=y_k),\mathcal{F}^Y_{t-1}], \quad \text{Assumption 6}
\end{aligned}$$

the last line holds as the future assignments are not informed by the historical ones. Applying this result twice gives the first result.

The second result follows by Adam's law and using the assumed independence of $W_{k,t}$ from all the other assignments.