CASE 6

What Is *Pi*, Anyway?

PRE-CASE EXERCISES

Please complete the following:

1. Construct a regular hexagon which is inscribed in a circle of radius 1" and compute its area and circumference (you may find it useful and interesting to use a computer program that sketches geometric figures).
2. Construct a regular hexagon which is circumscribed about the circle from #1 and compute its area and circumference.
3. How can you use the above calculations to approximate π as Archimedes did thousands of years ago? Is this a very good approximation?
4. What other methods can you use to get an even better estimate of π?

THE CASE

The 28 students in Mrs. Hamilton's third period Algebra II class sat expectantly awaiting the return of their tests. She handed them their papers facedown and watched as they felt for a sticker before turning the tests over to see their grades. Mrs. Hamilton always put a sticker on an "A" paper. Many students had done well on this test. Eleven of them felt a glossy patch as they ran their fingers across the tops of their tests. Justin smiled when he turned his paper over and saw a 99% written at the top.

Justin

Justin was an unusual student. He was a tall, friendly young man who preferred to sit up front where he could see the board clearly. He moved to the United States from China as a tenth grader and was placed in a sophomore math class. Because he had great difficulty following the work in English, he was transferred into a ninth-grade course, where he could keep up more easily. When his English improved dramatically during the year, he found the work too easy and wanted to move back up to be with his peers. He sat in on the higher-level course for the last quarter and successfully passed the final exam, qualifying to take junior math the following year. He proved to be one of the most capable students in Mrs. Hamilton's class. Justin was insightful and thoughtful in his approach to mathematics. He frequently remained after class to discuss concepts that intrigued him, often delving into problems in some depth. So, when they reached question #6 as they went over the test (see Figure 6.1), Mrs. Hamilton was surprised when Justin asked, "Why does $144\pi - 108\pi$ equal 36π instead of 36?"

"Well, think of them as like terms. If you had $144x - 108x$ that would equal $36x$. You don't just throw out the x. In this case, *pi* is like an x," Mrs. Hamilton responded.

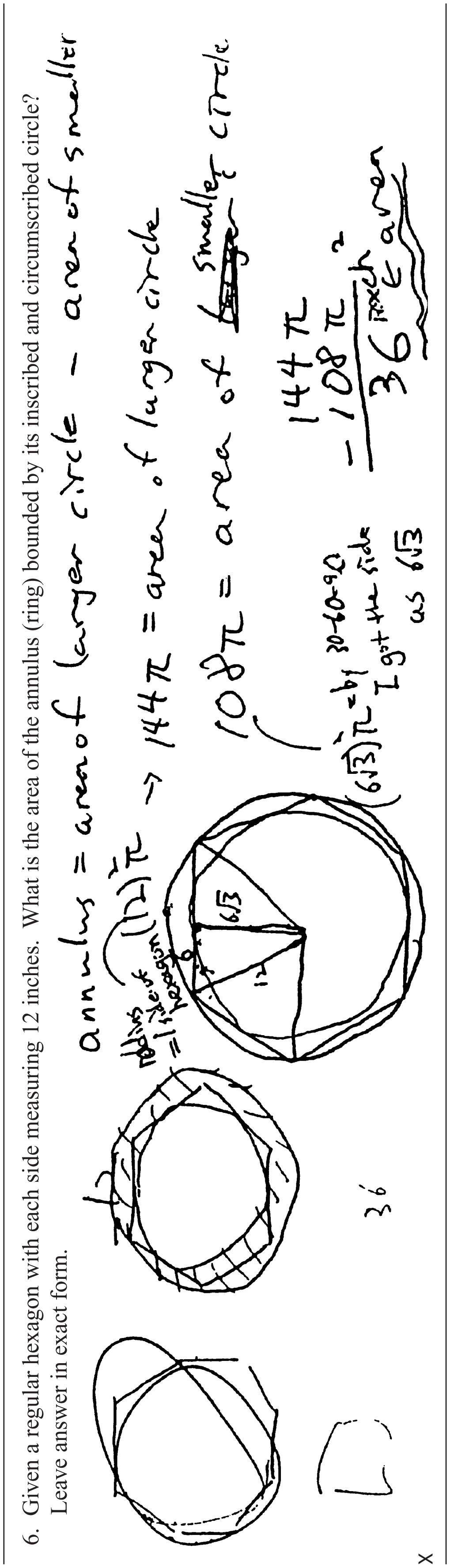
Justin looked unsure. Mrs. Hamilton watched him as he knit his brow and squinted at the board. "But how can there be a *pi* in the answer? You can only have *pi* if you are talking about a circle, and this is an annulus. It doesn't have a diameter!"

Mrs. Hamilton was puzzled. Justin had turned in a perfect paper except for this mistake. When she was grading the test she assumed that he had made a "careless" error and had simply forgotten to write down the π in his final answer and gave him five out of six possible points since she thought his error had been one of omission. Now it seemed that his mistake was due to a more profound misunderstanding. "But *pi* represents a number. It's like $\sqrt{2}$," she replied.

"I always thought *pi* was the ratio between the circumference and the diameter of a circle," Justin continued. "If you don't have a circumference and a diameter, how can you have *pi*?"

Mrs. Hamilton glanced around the room and

Figure 6.1 Question #6 on the Test



noticed that most of the other students had lost interest. They were talking among themselves about their homework and other concerns. Katie caught her eye and asked, "Can we go over number seven now?"

"You're free next period, right?" Mrs. Hamilton asked Justin. He nodded. "Great. Stay and I'll try to explain this to you."

Justin smiled and said, "Okay."

Mrs. Hamilton moved to problem #7 at the board.

Cookies

After class, Justin came up to Mrs. Hamilton's desk. She was gathering together the homework papers the students had just turned in. "I knew the *pi* should have been there if I followed the rules of algebra, but it just made no sense, so I left it away," Justin began.

"Let me see if I understand what you are saying," Mrs. Hamilton said as she put the papers in a folder and sat down. "You know how to find the area of a circle and you know that in order to solve this problem you had to find two areas and subtract them. I could see that from your work. But, you're confused about how you can have *pi* in the answer since an annulus doesn't have a circumference and a diameter. Is that right?"

"Yes. Well, I think it has a circumference—actually two of them—but it definitely doesn't have a diameter. And there's another thing I don't understand. How can the label be square units? Since this is a circle, shouldn't we use circle units or something?" Justin responded.

She paused for a moment. "Okay," she said. "Suppose you are making square cookies. You roll out the dough and you cut as many perfect square cookies as you can. When you first roll out the dough you don't get a perfect rectangle, so there's some left over. So, you mush all the extra dough together and roll it out again. Then you cut more cookies. Keep doing this until you run out of dough. If you count the number of cookies you cut out, then you know how many square units the original shape had in it."

"But wouldn't the cookies have some height? What if you rolled some out not as thick as others?" Justin asked.

"Well, actually, if you want that analogy to work, you have to think of the cookie dough like a plane. It has no thickness—just area. But I suppose you could think of it as really thin as long as you assume that you are perfect at rolling the dough exactly the same way every time," Mrs. Hamilton explained, beginning to think she may not have selected the best analogy.

Justin looked at his watch. "I have to go make up a Spanish quiz. I don't have any free time tomorrow. Can I come back in and talk on Monday after school?"

"Sure. I still want to get at your question about *pi* in the annulus. Would you mind making that your topic for this week's journal entry? That would help me understand where your confusion lies, which may make it easier for me to figure out a good way to explain it to you," Mrs. Hamilton responded.

"Okay," Justin said as he stuffed his book in his large, black backpack and hurried off down the hall.

The Journal Entry

After school on Monday, Mrs. Hamilton sat down at her desk and began to read the journal entry Justin had handed in earlier that day. This was the first year she had asked students to write them. She was finding the reflections interesting, but reading the journals each week was incredibly time-consuming. For that reason, Mrs. Hamilton was only having one class write them on a weekly basis. She asked the students to hand them in each Monday and required a minimum of two pages of reflections, questions, or independent mathematical exploration from each student. When she first announced that she would require journals, the students responded negatively. "We already do that in English!" one of the students had said. "But this is math! You can't write about math!" another added indignantly. However, Mrs. Hamilton had insisted. By third term most students seemed glad to have a place to express their thoughts and frustrations about math. Justin's entry read as follows:

Questions about math

It has been five years since I've learned about circles. During those five years I had no doubt that the area of a circle is πr^2. However, recently I've been questioning myself about the meaning of π and its relationship to the area.

I've learned that π is an infinite number. That it is Circumference/Diameter. I understand that the ratio of Circumference/Diameter would always, or very closely, would be the same, regardless of the size of the circle. The part that I did not understand was how an area could have an infinite area. Squares, rectangles, triangles, every polygonal region we are able to find the exact area. Circles are the only exception of what I've learned so far.

I found myself having questions when the equation of the circle was applied in problems, such as #6 on our recent test. The question was to find the area of an annulus (doughnut). As it happened, so many times throughout my short math career, the equation popped out of my mind automatically. Area of larger circle – area of smaller circle inscribed = annulus. So I did the problem without hesitation. But I started to doubt that it was the right answer. The answer as 36 π inch^2. The reason I thought it was wrong was because the π used in the area of an annulus. I knew by instinct that the equation I used was not wrong. But π in the area of an annulus? π is Circumference/ Diameter. An annulus may have a circumference once but it doesn't have a diameter or a radius, which I think are the properties of a circle that make us use π in our equations. I knew that I should've wrote the π in my answer, but it just wasn't convincing enough to write a symbol in a figure that didn't have the properties of the symbols. So I didn't use π in my answer even though I knew I was going to get it wrong.

QUESTIONS

1. What would you say to Justin about square units? What do you think of the cookie analogy?
2. One week later Justin took a quiz on which he had to find the volume of the material used to make a hollow ball with outer radius 6 cm and inner radius 4 cm. Justin wrote: $\frac{4}{3}$ (216) $\pi - \frac{4}{3}$ (64) π = 202 $\frac{2}{3}$ cm^3. What would you say to Justin?
3. How do you determine partial credit on a test? Should Justin have received less credit for the question since his error appeared to be more than "careless"?
4. Should Mrs. Hamilton have continued discussing the problem with the whole class instead of inviting Justin back later? How does Mrs. Hamilton know other students don't have the same misunderstanding? How could she find out?